2011 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value; however ties for individual awards will be broken based on problem difficulty (#11-20 are considered more difficult and will be used to break ties).

1. If $x^4 - 1 = 624$, find the largest possible real value of $x^3 + 107x^2 - 317x + 157$.

2. Find the degree of the following polynomial: $48x^2\left(3x(x+5)^3 + 6x^3\right)\left(x^2 - 7\right)^{11} + 44x^9 - 179$.

3. You are given 25 coins that look identical; however one of the coins is heavier than the others (the other 24 all have the same weight). You also have a pan balance that may be used to compare the weights of any two stacks of coins. For example, if you place stacks of 4 coins on each side of the balance and none of these was the heavy coin, then neither side of the balance will go down. Find the minimum number of weighings needed to be sure of finding the heavy coin.

4. The hypotenuse of a right triangle has length 37. The length of the longest leg is 35. Consider the circumscribed circle. Find the length of the arc subtended by the smallest angle of the triangle. Express your answer as decimal rounded to four significant digits.

5. ABCD is a rectangle with $AB = 45$ and $BC = k$. Let $i = \sqrt{-1}$. If $|45 + ki| = 53$, find the perimeter of the rectangle.

6. The points $(-6,15)$, $(9,15)$, and $(-6,7)$ are the vertices of a triangle. Rounded to the nearest whole degree, find the degree measure of the smallest angle of this triangle.

7. When $(2x-3)^{12}$ is expanded and completely simplified, find the numerical coefficient of x^5.

8. Let ABC be a triangle and suppose that the lengths of sides $\overline{AB}$ and $\overline{BC}$ are 20 and 80 meters, respectively. If $\angle ABC$ measures 120°, find the **exact** length of $\overline{AC}$.

9. If the speed of the current remains constant at 5.5 mph, a boat can travel 246 miles downstream in the same time it can travel 180 miles upstream. Find the speed (in mph) of the boat in still water. Express your answer as a decimal.

10. Urn A contains two orange marbles and one blue marble. Urn B contains two orange marbles and two blue marbles. An urn is selected at random and then one of the marbles is selected at random from that urn. Find the probability that the marble selected is orange. Express your answer as a common fraction reduced to lowest terms.

11. A committee of four is to be chosen randomly from five Whigs and six Tories. What is the probability that there is at least one person from each party on the committee? Express your answer as a fraction reduced to lowest terms.

12. A circle has the line $x - 2y + 4 = 0$ tangent at the point (0, 2) and $y = 2x - 7$ tangent at $(3,-1)$. Find the exact area of this circle.

13. When 2^{2011} is written as an integer, how many digits are there in the integer?

14. A woman is sitting in the stands behind one end zone of a football field. She observes, in the same vertical plane with her position, two players standing on their own goal lines, exactly 100 yards apart. Looking at the spots on the ground where they are standing, a player at one goal line is at an angle of depression of 15° and a player at the other goal line is at an angle of depression of 8°. Assuming the football field is horizontal, find the number of feet in the vertical height of the woman's eye above the horizontal plane of the football field. Express your answer as a decimal rounded to the nearest hundredth of a foot.

15. Find the smallest positive integer that has exactly 44 distinct positive divisors.

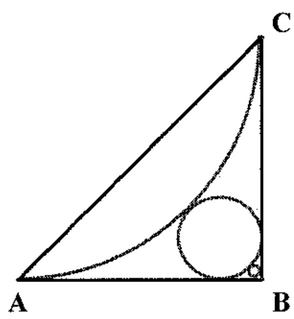
16. Let x and n be two-digit positive numbers (for which the tens digits are not zero). Suppose $n > x$. Find the value of x for which the sum of consecutive positive integers from 1 to $(x-1)$ is the same as the sum of consecutive positive integers from $(x+1)$ through n.

17. $\overline{AB}$ is the hypotenuse of ΔABC and has a length AB = 20. The radius of the inscribed circle has a length of 4. Find the perimeter of ΔABC.

18. The first term of an infinite geometric sequence of real terms is 225, and the fourth term of this geometric sequence is 14.4. Find the sum of the terms of this infinite geometric sequence.

19. Tom is a gambler. He has a pair of fair, standard cubical dice. He selects two different numbers in advance from the face numbers 1, 2, 3, 4, 5, and 6. He then rolls the pair of dice. If both numbers showing match his two selections, he wins \$108. If exactly one number showing matches one of his two selections, he neither wins nor loses. If neither number showing matches either of his two selections, he loses \$54. Find Tom's mathematical expectation (in dollars) each time he rolls the pair of dice. Express your answer by including the word "lose" or the word "win." For example, your answer might be "lose 2" or it might be "win 7."

20. Triangle ABC has sides $AB = BC = 1$ and angle B is a right angle. A large circle is tangent to AB at A and tangent to BC at C. A second circle is tangent to the large circle and also tangent to sides AB and BC. A third circle is tangent to the second circle and also tangent to sides AB and BC. If an infinite sequence of such circles is constructed, each tangent to AB, BC, and the previous circle, the sum of the areas of all of these circles is of the form $\left(\frac{x+y\sqrt{2}}{8}\right)\pi$. Find the ordered pair (x,y).



Name: ______ANSWERS______ Team Code: _______

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1. 4292
2. 28
3. 3
4. 12.22
5. 146
6. 28
7. -55427328
8. $20\sqrt{21}$
9. 35.5
10. 7/12
11. 31/33
12. 5π
13. 606
14. 88.67
15. 15360
16. 35
17. 48
18. 375
19. Lose 18
20. (4,3)