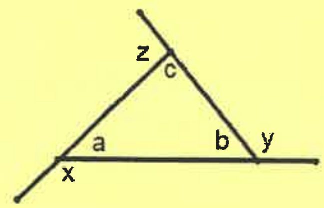
2019 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. For all real values of x, $6x^2+bx+c=3\left(ax^2+3x+4\right)$. Determine the sum $\left(a+b+c\right)$.

2. The sum of the mode and median is k larger than the arithmetic mean for the set of numbers $\{2, 2, 8, 2, 0, 1, 5\}$. Determine the value of k. Express your answer as a common or improper fraction reduced to lowest terms.

3. A triangle with sides extended is shown (but not necessarily drawn to scale). It has angle measures as labeled. $a:b:c=5:7:8$. Determine the ratio of $x:y:z$. Express your answer in the form $x:y:z$.

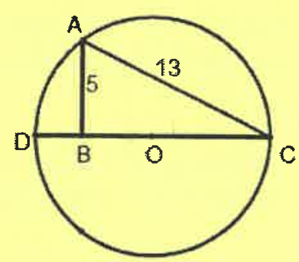


4. A prime number less than 20 is randomly selected. Determine the probability that the number selected is even. Express your answer as a common fraction reduced to lowest terms.

5. The slope of a ramp has a 5:12 ratio for vertical height to horizontal length measure. The length of the actual ramp is 39 feet. Determine the height of the ramp in feet.

6. A town's population increased by 700 people, and then this new population decreased by 5%. The town now has 20 fewer people than it did before the 700 person increase. Determine the number of people in the town's population.

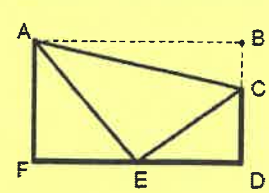
7. Points A, D, and C lie on circle O with diameter $\overline{CD}$. $\overline{AB}$ is perpendicular to $\overline{CD}$ at point B with $AB=5$ and $AC=13$. Determine the length of the diameter of this circle. Express your answer as a common or improper fraction reduced to lowest terms.



8. Determine the smallest integer x such that $\frac{1}{2}$ of $\frac{2}{3}$ of the sum of x and 6 is greater than $\frac{1}{6}$ of the sum of x and 30.

9. x is a positive odd integer. Determine *the sum* of all the distinct value(s) of x such that $-\left(x-11\right)=\left|x-11\right|$.

10. A triangular piece broke off rectangle $ABDF$ leaving trapezoid $ACDF$. $BD=16$, $BC=7$, $FD=24$, and E is the midpoint of $\overline{FD}$. Find the perimeter of ΔACE.



Name: ____________________ Team Code: ______

2019 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________

6. ____________________

7. ____________________

8. ____________________

9. ____________________

10. ____________________

11. ____________________

12. ____________________

13. ____________________

14. ____________________

15. ____________________

16. ____________________

17. ____________________

18. ____________________

19. ____________________

20. ____________________

Name: ______ANSWERS______ Team Code: ________

2019 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 23
2. $\frac{8}{7}$ Must be this reduced fraction.
3. 15:13:12 Must be exactly this format.
4. $\frac{1}{8}$ Must be this reduced fraction.
5. 15 (feet optional)
6. 13,680 (people optional)
7. $\frac{169}{12}$ Must be this improper fraction.
8. 19
9. 36
10. 60
11. 37.5 (mph optional)
12. 100.5 Must be this exact decimal.
13. 27
14. 171 (feet optional)
15. $\frac{40}{49}$ Must be this reduced fraction.
16. 75
17. $\frac{1}{2}$ Must be this reduced fraction.
18. 17
19. 2.580 Must be this exact decimal (trailing zero is necessary).
20. (6,8) Must be this ordered pair.

Awards Lists and Solutions to the Team Competition may be found at http://math.nku.edu/job

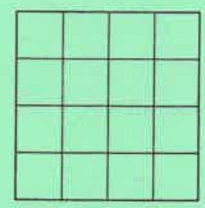
2019 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. $3^{x-2y} = \frac{3^{2x-y}}{81}$. Determine the sum $(x + y)$.

2. Determine the value of k so that $3x^2 - 6x + k = 12$ has exactly one double root.

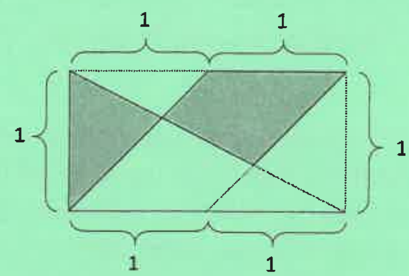
3. Determine the number of distinct prime factors for the number 426,888.



4. Determine the number of rectangles in the figure on the right.

5. Jordan drove to a town 25 miles away at an average speed of 50 miles per hour. The return trip along the same route took 20 minutes longer than the trip to town. Determine the average speed in miles per hour for Jordan's round trip. Express your answer as an exact decimal.

6. Determine the total area of the shaded region in the rectangle. Express your answer as a common or improper fraction reduced to lowest terms.



7. Determine the exact distance from the center of the circle given by $x^2 + 8x + y^2 - 6y + 3 = 2$ to the vertex of the parabola given by $y = x^2 + 4x + 3$. Express your answer in the form $a\sqrt{b}$ where a and b are integers and a+b is as small as possible.

8. A right circular cone with base diameter 10 and slant height 13 fits inside a prism with a square base. The base of the cone is inscribed in the base of the prism and both the cone and the prism have the same height. Determine the volume interior to the prism but exterior to the cone. Round your answer to four significant digits.

9. Given $g(x) = 6^{0.6x}$. Determine $g^{-1}(36)$. Express your answer as a common or improper fraction reduced to lowest terms.

10. Suppose $x = \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}$. Determine the exact value of x.

Name: ______________________ Team Code: _______

2019 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ______________________ 11. ______________________

2. ______________________ 12. ______________________

3. ______________________ 13. ______________________

4. ______________________ 14. ______________________

5. ______________________ 15. ______________________

6. ______________________ 16. ______________________

7. ______________________ 17. ______________________

8. ______________________ 18. ______________________

9. ______________________ 19. ______________________

10. ______________________ 20. ______________________

Name: ______ANSWERS______ Team Code: ________

2019 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 4

2. 15

3. 4

4. 100

5. 37.5 (miles per hour optional units)

6. $\frac{5}{6}$ Must be this reduced fraction.

7. $2\sqrt{5}$ Must be this simplified form.

8. 885.8 Must be exactly this answer.

9. $\frac{10}{3}$ Must be this simplified form.

10. 5

11. $\frac{1}{2}$ Must be this reduced fraction.

12. $(3,1)$ Must be this ordered pair.

13. $10\sqrt{59}$ Must be this simplified form.

14. $\frac{6}{5}$ Must be this reduced fraction.

15. 3

16. 58.73 Must be exactly this answer.

17. 2.58 Must be exactly this answer.

18. 144

19. 48

20. 7

2019 John O'Bryan Mathematical Competition
Questions for the Two-Person Speed Event

Calculators may not be used on the first four questions

1. Let $k = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}$ with $a + b + c = 15$ and $abc = 5$. Let $w = \frac{z}{xy} + \frac{y}{xz} + \frac{x}{yz}$ with $x^2 + y^2 + z^2 = 12$ and $xyz = 3$. Determine $(k + w)$.

2. Let $f(x) = sin^2(19x) + cos^2(19x)$ and $g(x) = \begin{vmatrix} 2 & x \\ -1 & 9 \end{vmatrix}$. If $k = f(g(2))$ and $w = g(f(2))$, then determine $(k + w)$.

3. Let $\log_8 64 + \log_2 k - 2\log_4 8 = 4$ and $3\sin\frac{\pi}{6} + 4\tan\theta\pi + 7\cos\frac{5\pi}{3} = 1$ with $0 \le \theta \le 1$. Determine $\frac{k}{\theta}$.

4. Let s be the coefficient of x^4 in $(x + 1)^7$ and t be the coefficient of y^3 in $(y + 2)^5$. Determine $(s - t)$.

Calculators may be used on the remaining questions

5. Let a be a positive integer number base so that $28_a = 132_5$. Let $3^{b-2c} = \frac{3^{2b-c}}{81}$. Determine $(a + b + c)$.

6. A circle C has a diameter with endpoints $A(4, -1)$ and $B(2,5)$. If C has center (h, k) and radius r, determine $h + k + r^2$.

7. Let p be the probability of rolling at least one 6 on one roll of two fair, standard, cubical dice. Let q be the probability of rolling a sum of 20 on one roll of two fair, standard, 20-sided dice. Determine (pq). Provide your answer as a common reduced fraction.

8. Let $k = \frac{1}{\sqrt{1968}} + \frac{1}{\sqrt{1969}} + \frac{1}{\sqrt{1970}} + \cdots + \frac{1}{\sqrt{2019}}$. Let s be $\frac{1}{3}$ the length of the side of a cube whose surface area is 2019. Round $(5k + s)$ to the closest integer.

9. (Tiebreaker #1) Let $f(x) = \frac{x^2-x-12}{(x+2)(x^2-2x-15)}$ where s denotes the sum of z where $f(z) = 0$, and p denotes the product of all a where $x = a$ is a vertical asymptote. Determine $(s + p)$.

10. (Tiebreaker #2) Let p be the value of the product $(3x - 3)(3x - 6)(3x - 9) \cdots (3x - 60)$ when $x = 15$. Let z be the sum of the zeros of $f(x) = x^3 - x^2 - 6x$. Determine $(p + z)$.

Names: ______________________ School: ____________________

2019 John O'Bryan Mathematical Competition
Answers for the Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value; however ties for individual awards will be broken based on problem difficulty.

Calculators are not allowed to be used on the first four questions!

		SCORE
1.	______________________	_____
2.	______________________	_____
3.	______________________	_____
4.	______________________	_____
5.	______________________	_____
6.	______________________	_____
7.	______________________	_____
8.	______________________	_____
T1.	______________________	
T2.	______________________	

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE

Name: ______ANSWERS______ **Team Code: _______**

2019 John O'Bryan Mathematical Competition
Answers for the Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value; however ties for individual awards will be broken based on problem difficulty.

Calculators are not allowed to be used on the first four questions!

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

1. 7

2. 20

3. 24

4. -5

5. 21

6. 15

7. $\frac{209}{14,400}$

8. 12

T1. -6

T2. 1

2019 John O'Bryan Mathematics Competition :: 5-Person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper.** Questions will not be scored without the following two things:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. Teams must **show complete solutions (not just answers)** to receive credit. More specific instructions are read verbally at the beginning of the test.

1. Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_6x^6$ be an arbitrary polynomial of degree 6. Define
$$g(x) = xf(x) - 1.$$
   (a) Find $g(0)$ and $g(1)$. Note: your answer may include the constant coefficients $a_0, a_1, a_2, \ldots, a_6$.
   (b) If $f(n) = \frac{1}{n}$ for $n = 1, 2, \ldots, 7$, find $g(n)$ for $n = 1, 2, \ldots, 7$.
   (c) Continuing from part (b), find $f(8)$. (Hint: $f(8) \neq \frac{1}{8}$.)

2. A non-negative integer m is a *square number* if $m = x^2$ for some integer x.
   (a) Determine how many positive integers n with $n \leq 20$ can be written as the sum of two square numbers (not necessarily distinct). That is, $n = a^2 + b^2$ for some integers a and b.
   (b) Assume $n = a^2 + b^2 + c^2$ with $a \geq b \geq c > 0$. Note that $a^2 + b^2 - c^2 > 0$. Prove that n^2 is also the sum of three square numbers.

3. Let $S(x)$ be the function that sums the digits of a positive integer x; e.g. $S(254) = 2 + 5 + 4 = 11$.
   (a) Determine $x + S(x) + S(S(x))$ when $x = 2019$.
   (b) Consider M where $M = x + S(x) + S(S(x))$ for some positive integer x. Prove that M is a multiple of K, where K is some positive integer.

4. Let A, B, C be vertices of a triangle with angles a, b, c (see Figure 1). Let O be the center of the circle outside of $\triangle ABC$ tangent to $\overline{BC}$, $\overrightarrow{AB}$, and $\overrightarrow{AC}$. Note that O is the intersection of the bisector of the angle A, and the bisector of the exterior angle at vertex B.
   (a) Determine $\angle AOB$ in terms of c.
   (b) Let D be the center of a circle that passes through points A, B, and O. Determine $\angle ADB$.
   (c) Prove that A, B, C, and D lie on a circle.

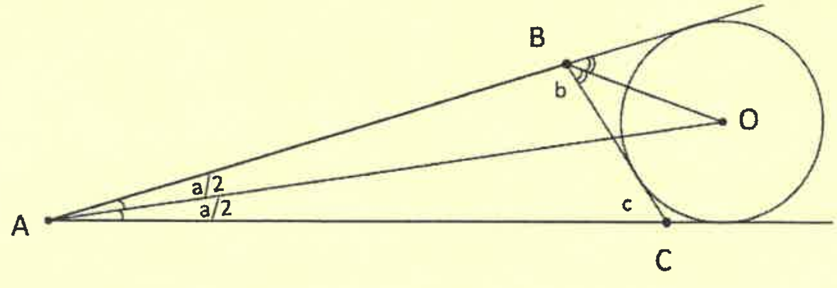


Figure 1: Scenario described in Question 4

2019 John O'Bryan Mathematics Competition :: 5-Person Team Test

1. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_6 x^6$ be an arbitrary polynomial of degree 6. Define

$$g(x) = xf(x) - 1.$$

(a) Find $g(0)$ and $g(1)$. Note: your answer may include the constant coefficients $a_0, a_1, a_2, \ldots, a_6$.

Solution: By definition,

$$g(0) = 0 \cdot f(0) - 1 = -1,$$

and

$$g(1) = 1 \cdot f(1) - 1 = (a_0 + a_1 + a_2 + \cdots + a_6) - 1.$$

(b) If $f(n) = \frac{1}{n}$ for $n = 1, 2, \ldots, 7$, find $g(n)$ for $n = 1, 2, \ldots, 7$.

Solution: By definition and assumption, for $n = 1, 2, \ldots, 7$,

$$g(n) = nf(n) - 1 = n \cdot \frac{1}{n} - 1 = 0.$$

(c) Continuing from part (b), find $f(8)$. (Hint: $f(8) \neq \frac{1}{8}$.)

Solution: Note that g is a 7^{th} degree polynomial with roots at $n = 1, 2, \ldots 7$. So $g(x) = A(x-1)(x-2)\cdots(x-7)$. Thus

$$g(0) = A(-1)(-2)\cdots(-7) = -A \cdot 7! = -1,$$

where the last equality follows from part (a). Then $A = \frac{1}{7!}$, making

$$g(x) = \frac{1}{7!}(x-1)(x-2)\cdots(x-7).$$

Hence $g(8) = \frac{1}{7!}(7 \cdot 6 \cdots 1) = 1$. Using the definition of $g(x)$, we have

$$1 = g(8) = 8f(8) - 1,$$

which gives $f(8) = \frac{1}{4}$.

2. A non-negative integer m is a *square number* if $m = x^2$ for some integer x.

(a) Determine how many positive integers n with $n \leq 20$ can be written as the sum of two square numbers (not necessarily distinct). That is, $n = a^2 + b^2$ for some integers a and b.

Solution: Note that 0, 1, 4, 9, and 16 are the squares that are at most 20. There are $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$ possible sums of distinct squares (only one of which is more than 20), and 5 possible sums of the same square (two of which are not in the interval $[1, 20]$). This gives a maximum of $9 + 3 = 12$ sums of two squares. Listing the possible positive sums to check for repeats gives

$$1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20.$$

Therefore, there are 12 such positive integers.

(b) Assume $n = a^2 + b^2 + c^2$ with $a \geq b \geq c > 0$. Note that $a^2 + b^2 - c^2 > 0$. Prove that n^2 is the sum of three, positive square numbers.

Solution: Let $n = a^2 + b^2 + c^2$. Expanding and rearranging gives

$$\begin{aligned}
(a^2 + b^2 + c^2)^2 &= a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 \\
&= \left(a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2\right) + 4a^2c^2 + 4b^2c^2 \\
&= \left(a^2 + b^2 - c^2\right)^2 + (2ac)^2 + (2bc)^2,
\end{aligned}$$

which is the sum of three squares.

3. Let $S(x)$ be the function that sums the digits of a positive integer x; e.g. $S(254) = 2 + 5 + 4 = 11$.

   (a) Determine $x + S(x) + S(S(x))$ when $x = 2019$.

   > **Solution:** Note $S(2019) = 2 + 0 + 1 + 9 = 12$. Then $2019 + S(2019) + S(S(2019)) = 2019 + 12 + 3 = 2034$.

   (b) Consider M where $M = x + S(x) + S(S(x))$ for some positive integer x. Prove that M is a multiple of an integer K with $K > 1$.

   > **Solution:** Note that a positive integer is divisible by 3 exactly when the sum of its digits is divisible by 3. Thus $x \equiv S(x) \equiv S(S(x)) \mod 3$. Hence, $x + S(x) + S(S(x)) \equiv 3x \equiv 0 \mod 3$. Therefore, M must be a multiple of 3.

4. Let A, B, C be vertices of a triangle with angles a, b, c (see Figure 1). Let O be the center of the circle outside of $\triangle ABC$ tangent to $\overline{BC}$, $\overrightarrow{AB}$, and $\overrightarrow{AC}$. Note that O is the intersection of the bisector of the angle A, and the bisector of the exterior angle at vertex B.

(a) Determine $\angle AOB$ in terms of c.

Solution: Notice that $\angle BAO = \frac{a}{2}$, $\angle CBO = \frac{180^\circ - b}{2}$ and $\angle ABO = b + \angle CBO = 90^\circ + \frac{b}{2}$. From the angle sum for $\triangle AOB$ and $\triangle ABC$ we have

$$\angle AOB = 180^\circ - \frac{a}{2} - \left(90^\circ + \frac{b}{2}\right) = 90^\circ - \frac{a}{2} - \frac{b}{2} = \frac{180^\circ - a - b}{2} = \frac{c}{2}.$$

(b) Let D be the center of a circle that passes through points A, B, and O. Determine $\angle ADB$.

Solution: Recall that the *inscribed angle theorem* states that for fixed points A and B on a circle with center P, an inscribed angle $\angle AMB$ is $\frac{1}{2}$ the central angle $\angle APB$. Notice that $\angle ADB$ is a central angle for the circle centered at D and intersecting A, B, and O. Also notice that $\angle AOB$ is an inscribed angle in the same circle. By the inscribed angle theorem, $\angle ADB = 2\angle AOB = 2 \cdot \frac{c}{2} = c$

(c) Prove that A, B, C, and D lie on a circle.

Solution: Let M be the center of a circle intersecting A, B, and C. Since $\overline{AB}$ is a chord of this circle, M must lie on the perpendicular bisector of $\overline{AB}$. Since $\angle ACB = c$, the inscribed angle theorem gives that $\angle AMB = 2c$. Since $\angle MAB = \angle MBA$, the degree sum of triangles gives $\angle MAB = \frac{180-2c}{2} = 90 - c$. Thus there is a unique location for M at the point of intersection of the perpendicular bisector of $\overline{AB}$ and the ray from A that creates an angle of $(90 - c)$ with $\overline{AB}$.

Since $\angle ADB = \angle ACB$ by part (b), the exact same argument holds for the circle with center M' intersecting A, B, and D. This means $M = M'$ and implies that the two circles are identical. Therefore, A, B, C, and D lie on a circle.

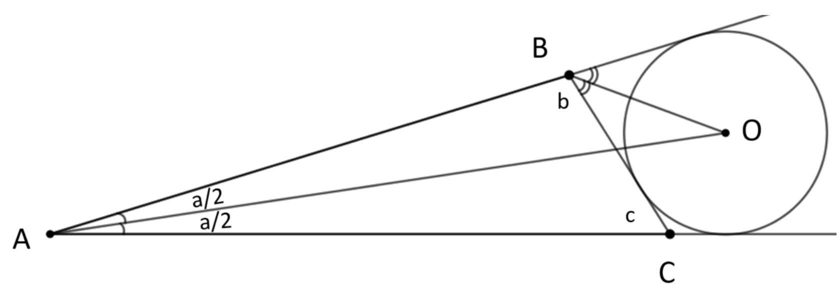


Figure 1: Scenario described in Question 4

5. At the start of the school year Jamie has 18 classmates in maths class, for a total of 19 students in the class. Each of Jamie's 18 classmates has a unique number of friends in the class (which could include Jamie). For example, Alex and Riley can **not** both have 7 friends in the class.

   (a) Let M be the maximum number of friends one of Jamie's classmates could have, and m be the minimum number of friends one of Jamie's classmates could have. What is M? What is m? Can there be a classmate with M friends, and a classmate with m friends at the same time?

   Solution: A classmate could be friends with everyone else in the class, so $M = 18$. A classmate could have no friends, so $m = 0$. No, if there is a classmate who is friends with everyone else, then no one can have no friends. Further, if there is a classmate with no friends, then no one can be friends with everyone.

   (b) Let ℓ denote the least number of friends any of Jamie's classmates has, and let p denote the most number of friends any of Jamie's classmates has. List all possible ordered pairs (ℓ, p) for Jamie's class. For each ordered pair, suppose that both Jamie's classmate with the least number of friends and Jamie's classmate with the most number of friends move away during the school year; what would be the new ordered pair?

   Solution: Since $M = 18$ and $m = 0$, there are 19 possible numbers of friends for each classmate. By part (a), M and m can not both occur, which means the ranges for number of friends is either 0 to 17, or 1 to 18. Thus there are only 18 possible numbers of friends for each classmate. Since all 18 classmates have a different number of friends, all of the 18 possible numbers of friends is taken. Therefore $(0, 17)$ and $(1, 18)$ are all possible ordered pairs.

   For $(0, 17)$ notice that the classmate with the most friends is friends with everyone except for the classmate with no friends. For $(1, 18)$ notice that the student with 1 friend is only friends with the classmate with the most friends. In either case when the two classmates move away, every classmate left in the class loses exactly one friend. Thus $(0, 17)$ would become $(1 - 1, 16 - 1) = (0, 15)$ and $(1, 18)$ would become $(2 - 1, 17 - 1) = (1, 16)$.

   (c) How many friends does Jamie have in maths class at the start of the school year?

   Solution: For both ordered pairs, Jamie's classmate with the least number of friends is not one of Jamie's friends, but Jamie's classmate with the most number of friends is one of Jamie's friends. If they both move away, then the same claim holds for the new pair of Jamie's classmates with the least and most number of friends. If we count Jamie's friends as we sequentially remove Jamie's two classmates with the least and most number of friends, we remove 9 pairs of classmates with one of the two being one of Jame's friends. Thus, Jamie has 9 friends in maths class.

6. Consider the operation of *square-replacement*, in which a square is replaced with four equal-sized squares (i.e. splitting the square into quarters).

(a) Starting from the unit square, consider the iterative process of square-replacing every square. Let $\alpha(k)$ denote the sum of perimeters for *all* squares after k iterations. Determine $\alpha(k)$ for $k = 0, 1, 2, 3$ and general k.

(a) $k = 0$ (b) $k = 1$ (c) $k = 2$ (d) $k = 3$

Figure 2: Iterations of square replacement, having 1, 4, 16, and 64 squares, respectively.

Solution: After k iterations, there are $\left(2^k\right)^2$ squares each with side length 2^{-k}. Thus

$$\alpha(k) = 2^{2k}\left(4 \cdot 2^{-k}\right) = 2^{k+2}.$$

So $\alpha(0) = 4$, $\alpha(1) = 8$, $\alpha(2) = 16$, and $\alpha(3) = 32$.

(b) Continuing from part (a), let $\delta(k)$ denote the sum of perimeters of squares that overlap the main diagonal of the unit square (possibly at a corner) after k iterations. Determine $\delta(k)$ for $k = 0, 1, 2, 3$ and general k.

(a) $k = 0$ (b) $k = 1$ (c) $k = 2$ (d) $k = 3$

Figure 3: Squares contributing to $\delta(k)$, having 1, 4, 10, and 22 squares, respectively.

Solution: After k-iterations, there are 2^k squares whose diagonal lies on the main diagonal of the unit square and $2 \cdot \left(2^k - 1\right)$ squares that overlap the main diagonal only at a corner. The side lengths of these squares is still 2^{-k}. Thus

$$\delta(k) = \left(4 \cdot 2^{-k}\right)\left(2^k + 2(2^k - 1)\right) = 12 - 2^{3-k}.$$

So $\delta(0) = 4$, $\delta(1) = 8$, $\delta(2) = 10$, and $\delta(3) = 11$.

(c) Is it possible to square-replace in a way that $\delta(k) > 2019$ for some k?

Solution: Yes. Note that $\delta(k) < 12$ for square-replacing every square, which does not work. Square-replacing the unit square gives four squares. We call the two squares with sides $\frac{1}{2}$ and overlapping the main diagonal only at a corner level-1 squares, and then square-replace the remaining squares. Next, we call the four squares with sides $\frac{1}{4}$ and overlapping the main diagonal only at a corner level-2 squares, and then square-replace the remaining squares (leaving the level-1 squares in tact). We continue in this way until we have 505 levels of squares. There are 2^k squares at level k, each with side 2^{-k}. Thus, the total perimeter of all level-k squares is 4, and the total perimeter of all squares overlapping the diagonal is $4 \cdot 505 = 2020 > 2019$, as desired.

(a) $k = 0$ (b) $k = 1$ (c) $k = 2$ (d) $k = 3$