2016 John O'Bryan Mathematics Competition
5-Person Team Test

Abbreviated Instructions: Answer each of the following questions **using separate sheet(s) of paper for each numbered problem**. Place your team code in the upper right corner of each page that will be turned in (failure to do this will result in no score). Place problem numbers in the upper left corner. Problems are equally weighted; **teams must show complete solutions (not just answers) to receive credit**. More specific instructions are read verbally at the beginning of the test.

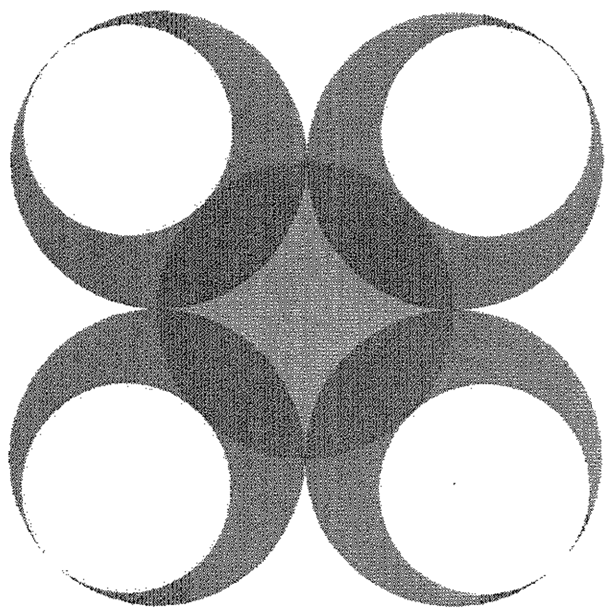
1. If $x^2+x+1=0$, then

   a. Show that $x^3=1$.

   b. Show that $\left(x+\frac{1}{x}\right)^2=1$.

   c. Find the value of $\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)^3+\left(x+\frac{1}{x}\right)^4+\left(x+\frac{1}{x}\right)^5$.

2. It has been your lifelong dream to paint this on your dorm room wall:



Notice that the design consists of nine circles: five of one large size, four of another smaller size. If the radius of each large circle is one foot, determine how much paint of each shade of gray is needed to paint this. Your answers should be in square feet. (Notice there are four different shades, varying from off white to full gray.)

3. Two ferry boats start at the same instant from opposite sides of a river with parallel shorelines, traveling across the water on routes at rights angles to the shore. Each boat travels at a constant speed, although one is faster than the other. They pass at a point 720 yards from the nearest shore, and upon reaching the opposite shore, each boat docks for exactly 10 minutes. After docking for 10 minutes, each boat heads back to its original starting position, retracing its original path, and this time the boats pass at a point which is 400 yards from the nearest shore. How wide is the river?

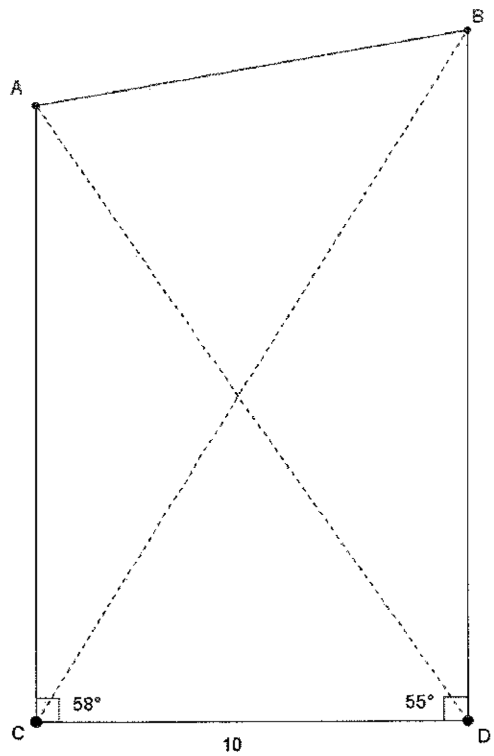
4. Seven coins weighing 1, 2, 3, 4, 5, 6, and 7 grams are given, but which weighs how much is unknown. A man claims he knows exactly which coins are which, and as an offer of proof, he claims he can perform a single weighing on a balance scale so as to unequivocally demonstrate the weight of at least one of the coins. Is this possible or is he exaggerating?

5. It may sound glamorous to be a vampire, but there's a dark secret our nocturnal pals don't share outside the coven. If a vampire feeds twice in the same evening, and the two blood types do not match, he will get an upset stomach. He then won't be able to sleep all day. Here is the distribution of the eight blood types among people in the US.

Type	O+	O-	A+	A-	B+	B-	AB+	AB-
Percent	41	6	31	5	11	2	3	1

One night, Vinny the Vampire picks two people from the US at random and chows down. Determine the probability that he does not get sick. Blood types must match exactly: the notion of "universal donor," etc., does not apply here.

6. In the diagram, find the length of $\overline{AB}$, given that $\overline{CD}=10$, $m(\angle BCD)=58°$, $m(\angle ADC)=55°$, and $m(\angle ACD)=m(\angle BDC)=90°$.



Team Solutions

1.

a. The product is $x^3=1$. Then note that $x^2= 1/x$, $x=1/x^2$, $1= 1/x^3$

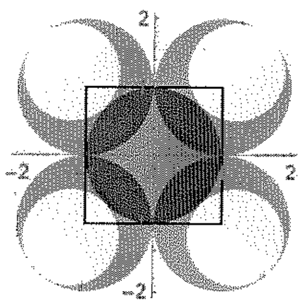
b. $(x+1/x)^2 = x^2+1/x^2+2 =(x^2+x+1)+1=1$

c. $(x+1/x)^3=(x+1/x)\ 1= x+x^2=(1+x+x^2)-1=-1$
$(x + 1/x)^4= ((x + 1/x)^2)^2=1^2=1$
$(x+1/x)^5= (x+1/x)^2 (x+1/x)^3 = 1 (-1) =-1$

So the sum must be 1+ (-1) + (1) + (-1) =0.

2.

Put in a coordinate system with the origin at the center of the middle large disk. Since it has radius 1, the points where the four other large disks are tangent to each other is a distance of 1 from the origin. This means the coordinates for the large disks are (1,1),(-1,1),(-1,-1),(1,-1). Draw the square with vertices at these centers.



To get the radius of a small circle:

- the distance from the origin to where the middle circle is tangent to a small circle is 1
- the distance from the origin to the center of a large outer circle is $\sqrt{2}$ since it is a diagonal of a unit square
- the distance from the center of a large outer circle to the point of tangency with a small circle inside it is 1 since it is the large circle's radius.

This means the diameter of a small circle is $\sqrt{2}+1-1=\sqrt{2}$ or the radius is $\frac{\sqrt{2}}{2}$.

1. The off-white paint needs to cover the area of the four small disks:
   $Area=4\left(\pi\left(\frac{\sqrt{2}}{2}\right)^2\right)=4\pi\frac{2}{4}=2\pi$.
2. The medium light gray paint needs to cover the area of the 2×2 square minus the area of 4 of a quarter of each of the large outer disks: $Area=2\times2-4\left(\frac{1}{4}\pi(1)^2\right)=4-\pi$.
3. The darkest gray paint needs to cover the area of the one middle disk minus the area covered by the medium light gray paint:
   $Area=\pi(1)^2-(4-\pi)=2\pi-4$.
4. The medium dark gray paint needs to cover each of the 4 large outer disks minus the area covered by the off-white paint minus the darkest areas: $Area=4\left(\pi(1)^2\right)-2\pi-(2\pi-4)=4$.

3.

Let v_1 be the velocity of the faster boat and v_2 the velocity of the slower boat.

From the first meeting we get : $t_1 v_2=720$ and $t_1 v_1= w-720$, where w is the width of the river.

From the second meeting we get: $(t_2-10)v_1=w+400$ and $(t_2-10)v_2=2w-400$.

Taking the ratio $v_1/v_2= (w+400)/(2w-400)= (w-720)/720$ or $w(w-1760) =0$. So the river is 1760 yards wide.

4.

Yes it is possible. Notice that 7+6 = 1+3+4+5. So by removing the 2, we have equal weights. But is this the only way to get equal weights using all but one of the weights First note that the unused coin must be even (so that the remainder is divisible by 2.)

Remove the 6: Note that the 7 and the 5 must be in different groups. Then we need to divide the coins 1,2,3,4 into two groups such that one sums to 2 more than the other.

So only 7,3,1 and 5,4,2.

Remove the 4: The only possible way to get equal weights is 1+2+3+6= 5+7.

So the demonstration is to put the 6 coin aside, and put the 7,3,1 and 5,4,2 coins on each side of the scale. This is the only solution using the same number of coins in each pan.

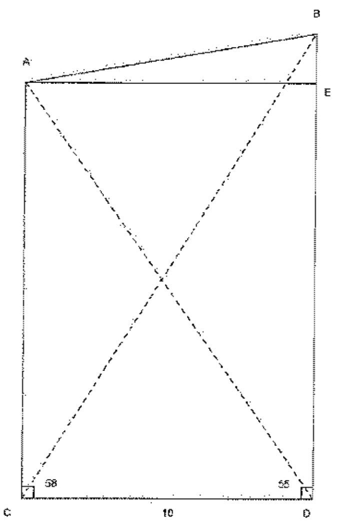
5.

Note that the probability of picking two with type O+ blood is $\frac{41}{100}\cdot\frac{40}{99}$. Using this logic for each of the types, the probability he does not get sick is

$$\frac{41}{100}\cdot\frac{40}{99}+\frac{6}{100}\cdot\frac{5}{99}+\frac{31}{100}\cdot\frac{30}{99}+\frac{5}{100}\cdot\frac{4}{99}+\frac{11}{100}\cdot\frac{10}{99}+\frac{2}{100}\cdot\frac{1}{99}+\frac{3}{100}\cdot\frac{2}{99}+\frac{1}{100}\cdot\frac{0}{99}=\frac{1369}{4950}\approx 0.2766$$

6.

Using trigonometry we get BD = 10 tan 58 and AC = 10 tan 55. Adding a horizontal from A to point E:



We have $AB^2=10^2+(BD-AC)^2$ or that $AB=10\sqrt{1+\tan 58-\tan 55}\approx 10.8268$.
This is approximately 10.8268.