Transition To College Mathematics

In Support of Kentucky's
College and Career Readiness Program

Northern Kentucky University
Kentucky Online Testing (KYOTE) Group

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Preface

College Readiness. What does it mean to be college ready in mathematics and how can it be measured? While the nation struggles to answer these questions, Kentucky alone among the states has answered both questions and is moving ahead with a bold new program to implement its statewide agreement on these answers. The learning outcomes students need to know to be considered college ready in mathematics and the passing scores on the assessments used to measure college readiness in Kentucky are given in Appendix 2.

The Kentucky college readiness program is based on the ACT assessment taken by all public high school juniors. Students who score at least 19 on the ACT math exam, at least 18 on the ACT English exam and at least 20 on the ACT reading exam are considered college ready by the Kentucky Department of Education (KDE) and the Council on Postsecondary Education (CPE), and are guaranteed placement into credit bearing courses without the need for remediation at any public college or university in the commonwealth.

High school juniors who do not meet these ACT benchmarks are given a second chance to become college ready. They are offered transitional courses in mathematics, English or reading in their senior year. They take either a Kentucky Online Testing (KYOTE) or COMPASS placement exam after completing the course. If they pass, they are considered college ready in that area by the KDE and the CPE, and are guaranteed placement into credit bearing courses in that area without the need for remediation at any public college or university in the commonwealth. They are also offered opportunities to become career ready.

The college and career readiness model has so far been a resounding success. In 2010, only 34% of Kentucky high school graduates were college or career ready. That percentage jumped to 38% in 2011 and then went up 24% to an impressive 47.2% in 2012. The KDE estimates that 5,400 more high school graduates became college or career ready in 2012 than in 2011, saving students and their parents over 5 million dollars in tuition costs.

The percentage of high school graduates who are college or career ready in each district and in each high school for 2010, 2011 and 2012 can be found at http://education.ky.gov, clicking on *School Report Cards*, selecting a school or district, and clicking on *Delivery Targets*.

Despite this success, daunting challenges remain in reaching the goal mandated by Senate Bill 1 of getting 67% of all Kentucky high school graduates college or career ready by 2015.

Book Features. To address these challenges and the requests from the Northern Kentucky high school math teachers, we offer the first draft of our free online textbook, *Transition to College Mathematics*, as one of many strategies that can be used to get more Kentucky students ready for college mathematics. Although we are working with Northern Kentucky high school math teachers on this project, we welcome other Kentucky math teachers who would like to use this free resource.

Teachers and students can access the book on our website http://kyote.nku.edu.

The book is designed to cover in depth the KYOTE college readiness and college algebra placement exam standards listed in Appendix 1. The book focuses entirely on the standards. Additional content is included only occasionally and only when it adds greater depth to the topics covered by the standards. Teachers can be certain that if they cover the content in the book, then their students will not be blindsided on a placement exam by material they have not seen. This is a feature that Northern Kentucky teachers requested and it is no doubt important to other Kentucky teachers who are teaching transitional mathematics courses.

The book serves primarily two groups of students: those students who should take the college readiness placement exam and those students who should take the college algebra placement exam. The KYOTE college readiness placement exam is intended for students who scored below 19 on the ACT math exam. The KYOTE college algebra placement exam is intended for students who scored 19, 20 or 21 on the ACT math exam. Students who score in this range are considered college ready in mathematics and are guaranteed placement into *some* credit bearing mathematics course without the need for remediation, but not necessarily college algebra.

The ACT recommends, and most Kentucky universities require, an ACT math exam score of 22 or more for placement into college algebra. Students with an ACT math exam score of 19, 20 or 21 have in the past come to college campuses hoping to enroll in college algebra, a course required for a wide range of college majors. These students discover to their dismay that they will need a developmental mathematics course before they can take college algebra.

The KYOTE program has helped to highlight and reduce this problem. Students in this predicament can now take a transitional course in their senior year followed by the KYOTE college algebra exam, and begin their college career with college algebra at any college or university in Kentucky if they pass.

The KYOTE placement exam standards covered in each section of the book are listed at the beginning of that section. This feature will help teachers target specific standards and topics where students are struggling. It will also help them design their transitional courses to focus on students intending to take the college readiness exam or on students intending to take the college algebra exam. If students intend to take the college readiness exam, for example, then teachers can focus on the 25 sections in the book that cover college readiness standards. They do not need to cover the 8 remaining sections devoted exclusively to college algebra standards.

The book is designed for students who have taken both Algebra I and Algebra II, but who have not mastered the content of these courses. We believe that an in depth review of this content is the best way to prepare these students for the rigors of college mathematics.

But the book is not a standard algebra text. We emphasize basic arithmetic and some elementary geometry, and their connections to algebra. We emphasize developing fluency with decimals, percentages and especially fractions, an area where we know our high school graduates are particularly weak. We believe that a thorough understanding of such concepts as greatest common factor and least common denominator in arithmetic will lead to a greater understanding about how these concepts are extended to algebra.

We emphasize applied arithmetic problems involving units, rates and proportions. We show how algebra can be used to understand and solve this class of problems in a few cases, and we seamlessly move into problems whose solution requires algebra.

Background. The impetus for writing this book came from a February 2012 KYOTE meeting of Northern Kentucky high school math teachers. The teachers were surveyed about the support they most needed for their transitional courses. Almost unanimously, they requested curriculum materials for the course. The need was most acute in smaller districts and high schools. But there was also great interest from teachers in the larger high schools and districts, even from teachers who had taught transitional courses in the past and had some materials already available.

Exercise sets, organized into 10 chapters, were written in the summer to cover the KYOTE placement exam standards. These exercise sets were put online on our website http://kyote.nku.edu in the fall so that teachers could access them, use them in their transitional courses and give us feedback about them. The exercise sets were put in Microsoft Word format at the request of teachers so that exercises could easily be copied and pasted into existing transitional course materials.

The book was written in the fall around the exercise sets. It includes illustrative examples and explanatory text that support each set of exercises.

Next Steps. The book represents only one phase of an ongoing project. We are seeking feedback from teachers about the book and how it might be improved to serve students more effectively. We are looking for additional exercises that would strengthen the book. We are particularly interested in lesson plans that teachers have found to be helpful in shedding light on the topics covered. One suggestion we have already received from teachers is to include formative multiple-choice assessments at the end of each chapter to help students prepare for the multiple-choice placement exams.

We will continue conducting regular meetings with Northern Kentucky high school mathematics teachers to solicit feedback and to discuss issues relating to transitional courses. We will also continue visiting teachers and students at individual high schools to discuss transitional courses in particular and the transition to college mathematics in general.

The exciting culminating phase of the project is an online homework system for the transitional course offered through the University of Kentucky (UK) Web Homework System (WHS) at www.mathclass.org. The KYOTE system is a part of this larger UK system that currently serves about 5,000 mathematics and Spanish students per semester at UK as well as other colleges, universities and high schools. The online homework problems will be linked to examples and explanations in the textbook that students can use if they are having difficulty solving a homework problem. The problems can also be linked to other resources including videos. Teachers can use the tools provided by UK to customize their courses and add content as desired.

The online homework transitional course will be offered free of charge to all Kentucky educational institutions as is the case with all KYOTE products.

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Chapter 1. Foundations of Algebra

1.1. Signed Numbers, Exponents and Order of Operations

KYOTE Standards: *CR 1, CR 2; CA 1*

Exponents

Multiplication can be viewed as repeated addition. For example, $3 \cdot 4$ can be viewed as the sum of 3 $4's$ or the sum of 4 $3's$. Thus

$$3 \cdot 4 = 4+4+4 = 12 \qquad or \qquad 3 \cdot 4 = 3+3+3+3 = 12$$

Exponents can be used to express repeated multiplication. For example, the symbol 3^4 is used to represent the product of 4 $3's$. Thus

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

The symbol 3^4 is read "3 to the fourth power" or simply "3 to the fourth." The number 3 is called the *base* and the number 4 is called the *exponent*. We note that the base and exponent cannot in general be interchanged without changing the exponential expression being computed. For example, $3^4 = 81$ is not equal to $4^3 = 4 \cdot 4 \cdot 4 = 64$.

Example 1. Calculate $(-2)^3 \cdot 7^2$

Solution. Using the definition of exponential expressions, we have

$$\begin{aligned} (-2)^3 \cdot 7^2 &= (-2) \cdot (-2) \cdot (-2) \cdot 7 \cdot 7 \\ &= (-8) \cdot 49 \\ &= -392 \end{aligned}$$

Order of Operations

The order in which operations are performed to calculate the value of an arithmetic expression is critically important. Let's look at a few examples to see why.

- What is the value of $7-3 \cdot 5$? Is it $(7-3) \cdot 5 = 4 \cdot 5 = 20$ or is it $7-(3 \cdot 5) = 7-15 = -8$?
- What is the value of $5-9-2$? Is it $(5-9)-2 = -4-2 = -6$ or is it $5-(9-2) = 5-7 = -2$?
- What is the value of $5 \cdot 2^3$? Is it $(5 \cdot 2)^3 = 10^3 = 1000$ or is it $5 \cdot (2^3) = 5 \cdot 8 = 40$?
- What is the value of $1+2^4$? Is it $(1+2)^4 = 3^4 = 81$ or is it $1+(2^4) = 1+16 = 17$?

It's clear that we need some agreement about the order in which mathematical expressions should be evaluated. This agreement is stated below.

Order of Operations Rule

To evaluate mathematical expressions, carry out the operations in the following order:

1. Perform operations within parentheses.
2. Perform all exponentiation operations.
3. Perform all multiplications and additions from left to right.
4. Perform all additions and subtractions from left to right.

A few examples should illustrate this rule. We strongly recommended that you do these calculations without a calculator to develop fluency with signed numbers and exponents as well as order of operations. You are then welcome to check your answer using a calculator. You will notice that all calculators are programmed to follow the order of operations rule.

Example 2. Simplify
(a) $-3-7+2$ **(b)** $-3-(7+2)$ **(c)** $36\div 2\cdot 3$ **(d)** $36\div(2\cdot 3)$

***Solution*. (a)** Since $-3-7+2$ involves only additions and subtractions, we perform all operations from left to right to obtain

$$-3-7+2=(-3-7)+2=-10+2=-8$$

(b) Since $-3-(7+2)$ involves parentheses, we perform the operations within parentheses first to obtain

$$-3-(7+2)=-3-9=-12$$

(c) Since $36\div 2\cdot 3$ involves only multiplications and divisions, we perform all operations from left to right to obtain

$$36\div 2\cdot 3=(36\div 2)\cdot 3=18\cdot 3=54$$

(d) Since $36\div(2\cdot 3)$ involves parentheses, we perform the operations within parentheses first to obtain

$$36\div(2\cdot 3)=36\div 6=6$$

Example 3. Simplify

(a) $-6-3\cdot 2^3+(-4)^2$ **(b)** $(-6-3)\cdot 2^3+(-4)^2$

***Solution*. (a)** We perform all exponentiations first, then all multiplications, and finally all additions and subtractions from left to right to obtain

$-6-3\cdot 2^3+(-4)^2=-6-3\cdot 8+16$	Calculate 2^3, $(-4)^2$
$=-6-24+16$	Calculate $3\cdot 8$
$=-30+16$	Calculate $-6-24$
$=-14$	Calculate $-30+16$

(b) We first calculate $-6-3$ in parentheses. We next perform all exponentiations, then all multiplications, and then additions and subtractions to obtain

$(-6-3)\cdot 2^3+(-4)^2=(-9)\cdot 2^3+(-4)^2$	Calculate $-6-3$
$=-9\cdot 8+16$	Calculate 2^3, $(-4)^2$
$=-72+16$	Calculate $-9\cdot 8$
$=-56$	Calculate $-72+16$

Example 4. Find the value of x^3-2x^2y-y when $x=-1$ and $y=-3$.

***Solution*.** We substitute $x=-1$ and $y=-3$ into x^3-2x^2y-y and follow the order of operations rule to obtain

$(-1)^3-2(-1)^2(-3)-(-3)$	Substitute $x=-1$ and $y=-3$ Into expression x^3-2x^2y-y
$=-1-2(1)(-3)+3$	Calculate $(-1)^3$, $(-1)^2$
$=-1+6+3$	Calculate $(-2)\cdot 1\cdot(-3)$
$=8$	Calculate $-1+6+3$

Exercise Set 1.1

Simplify without using a calculator.

1. $3-4-5$

2. $3-(4-5)$

3. $7+2\cdot 8$

4. $(7+2)\cdot 8$

5. $-2(6-9)-1$

6. $-2\cdot 6-9-1$

7. $-9(2-5)-7$

8. $-9\cdot 2-(5-7)$

9. $30\div 2\cdot 5$

10. $30\div(2\cdot 5)$

11. $(30\div 2)\cdot 5$

12. $6-8\div 2$

13. $(6-8)\div 2$

14. $6-(8\div 2)$

15. $48\div 8\div 2$

16. $48\div(8\div 2)$

17. $(48\div 8)\div 2$

18. $3\cdot 5-3\cdot 7$

19. $3(5-7)$

20. $3\cdot(5-3)\cdot 7$

Simplify without using a calculator.

21. 3^4

22. 5^3

23. 2^5

24. 7^2

25. $(-4)^2$

26. -4^2

27. $(-2)^3\cdot(-3)^2$

28. $-2^3\cdot(-3)^2$

29. $(2-7)^2$

30. 2^2-7^2

31. $(3-5)^3$

32. 3^3-5^3

33. $-7+3\cdot 2^2$

34. $-7+(3\cdot 2)^2$

35. $(-7+3)\cdot 2^2$

36. $8-2\cdot 3^2$

37. $(8-2)\cdot 3^2$

38. $8-(2\cdot 3)^2$

39. $20\div 2^2$

40. $(20\div 2)^2$

41. $3\cdot 5^2$

42. $(3\cdot 5)^2$

43. $3^2\cdot 5^2$

44. $(3+5)^2$

45. 3^2+5^2

46. $-6+5\cdot(-3)^2$

47. $(-6+5)\cdot(-3)^2$

48. $-6-5\cdot 3^2$

Find the value of the algebraic expression at the specified values of its variable or variables.

49. $x^2-2x-5;\ x=-3$

50. $4-x;\ x=-3$

51. $x-2(1-3x);\ x=-1$

52. $x^3-4x^2-5x;\ x=-1$

53. $(x-y)^2;\ x=1, y=-3$

54. $x-(y-x^2);\ x=-3,\ y=2$

55. $-x+3y^2;\ x=-6,\ y=-2$

56. $3+xy^2;\ x=-5, y=-2$

57. $(3+x)y^2;\ x=-5, y=-2$

58. $3+(xy)^2;\ x=-5, y=-2$

1.2. Prime Numbers, *GCF* and *LCM*

KYOTE Standards: *Foundational Content*

A positive integer can be factored in a several ways. For example, 12 can be factored in three ways as $12 = 1 \cdot 12$, $12 = 2 \cdot 6$ and $12 = 3 \cdot 4$. The factors of 12 are the six positive integers $1, 2, 3, 4, 6, 12$. Positive integers like 12 with more than two factors are called *composite* numbers. On the other hand, 7 can be factored in only one way as $7 = 1 \cdot 7$. The factors of 7 are the two positive integers $1, 7$. Positive integers like 7 with exactly two factors are called *prime* numbers. The formal definitions are given below.

Definition 1. A *prime number* is a positive integer that has exactly two factors, itself and 1. A *composite number* is a positive integer that has more than two factors.

Example 1. Find the number of factors of the given positive integer and determine whether that integer is prime or composite.

(a) 36 **(b)** 17 **(c)** 1 **(d)** 65

***Solution*. (a)** Note that $36 = 1 \cdot 36$, $36 = 2 \cdot 18$, $36 = 3 \cdot 12$, $36 = 4 \cdot 9$, $36 = 6 \cdot 6$. Thus 36 has nine factors $1, 2, 3, 4, 6, 9, 12, 18, 36$ and is a composite number.

(b) Note that $17 = 1 \cdot 17$ but no other factorizations. Thus 17 has two factors $1, 17$ and is a prime number.

(c) Note that $1 = 1 \cdot 1$ but no other factorizations. Thus 1 has only one factor and is therefore *not* a prime number.

(d) Note that $65 = 1 \cdot 65$, $65 = 5 \cdot 13$. Thus 65 has four factors and is a composite number.

The following result is an important building block in both arithmetic and algebra.

Fundamental Theorem of Arithmetic. Any positive integer greater than 1 can be written uniquely as the product of powers of prime numbers.

This product of powers of prime numbers is called the *prime factorization* of the positive integer. Let's consider a few examples. It is easy to check that the positive integer 12 can be written $12 = 2^2 \cdot 3$ and it is easy to find this factorization since 12 is a relatively small integer. It is easy to check that $924 = 2^2 \cdot 3 \cdot 7 \cdot 11$ with a calculator, but it is more difficult to find its prime factorization. Example 2 shows how to obtain the prime factorization of a relatively small positive integer by inspection and simple calculations. Example 3 shows how to obtain the prime factorization of a relatively large positive integer in a more deliberate fashion.

Example 2. Find the prime factorization of the given positive integer.

(a) 72 **(b)** 23 **(c)** 180

Solution. **(a)** We know that $72 = 8 \cdot 9$ and factorization of $8 = 2^3$ and $9 = 3^2$ gives the prime factorization $72 = 2^3 \cdot 3^2$.

(b) A search for factors of 23 yields only 23 and 1. Thus 23 is a prime number and its prime factorization is simply 23 itself.

(c) We see immediately that 10 is a factor of 180 and we write $180 = 18 \cdot 10$. We note that $18 = 2 \cdot 9 = 2 \cdot 3^2$ and $10 = 2 \cdot 5$ to obtain the prime factorization of $180 = 2^2 \cdot 3^2 \cdot 5$.

Example 3. Find the prime factorization of 6552.

Solution. The prime factorization of 6552 is more difficult to obtain but illustrates a general method of finding prime factorizations. We begin with the smallest prime number 2. Since 6552 is an even integer, we can divide it by 2 to obtain the factorization $6552 = 3276 \cdot 2$. We continue dividing by 2 until we have divided out all the 2's. We then divide out all the 3's, then all the 5's until we have divided out all the powers of the prime numbers and have obtained our prime factorization. We describe this process in detail for 6552.

$6552 = 3276 \cdot 2$	Divide 6552 by 2
$= 1638 \cdot 2 \cdot 2$	Divide 3276 by 2
$= 819 \cdot 2 \cdot 2 \cdot 2$	Divide 1638 by 2
$= 273 \cdot 3 \cdot 2 \cdot 2 \cdot 2$	Divide 819 by 3; note 819 cannot be divided by 2
$= 91 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2$	Divide 273 by 3
$= 13 \cdot 7 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2$	Divide 91 by 7; note that 91 cannot be divided by 3 or 5

Thus the prime factorization of 6552 is $6552 = 2^3 \cdot 3^2 \cdot 7 \cdot 13$.

Definition 2. The *greatest common factor* (*GCF*) of two positive integers a and b is the largest positive integer d that is a factor of both a and b. We write $d = GCF(a,b)$.

Definition 3. The *least common multiple* (*LCM*) of two positive integers a and b is the smallest positive integer m such that m is a multiple of both a and b. In other words, m is the smallest positive integer such that a and b are both factors of m. We write $m = LCM(a,b)$.

Note. You may not have heard the term "least common multiple" but you have certainly heard the term "least common denominator." The least common denominator used when adding two fractions is the least common multiple of the denominators of the two fractions.

To illustrate these definitions, let's determine the *GCF* and *LCM* of 12 and 18. The integers $1, 2, 3$ and 6 are all common factors of 12 and 18, and 6 is the largest. Thus $6 = GCF(12,18)$. The integer 36 is the smallest positive integer that is a multiple of both 12 and 18. In other words, both 12 and 18 are factors of 36 ($36 = 3 \cdot 12$ and $36 = 2 \cdot 18$) and no smaller integer has this property. Thus $36 = LCM(12,18)$.

Method for Finding *GCF* and *LCM*

There is a straightforward method to find the *GCF* and *LCM* of two positive integers a and b if we know their prime factorizations.

1. The *GCF* of a and b is the product of the largest power of each prime that is a factor of both a and b.

2. The *LCM* of a and b is the product of the largest power of each prime that is a factor of either a or b.

Example 4 shows how to use this method in some concrete cases.

Example 4. Find the greatest common factor and the least common multiple of the given pair of integers.

(a) $36,\ 40$ **(b)** $20,\ 27$ **(c)** $2^5 \cdot 3^2,\ 2^3 \cdot 3^4 \cdot 7$

Solution. **(a)** We consider the prime factorizations of 36 and 40:

$$36 = 2^2 \cdot 3^2 \qquad 40 = 2^3 \cdot 5$$

We see that 2^2 is the largest power of 2 that is a factor of both 36 and 40. No other prime is a factor of 36 and 40. Thus

$$GCF(36,40) = 2^2 = 4$$

We see that 2^3 is the largest power of 2 that is a factor of either 36 or 40, 3^2 is the largest power of 3 that is a factor of either 36 or 40, and that 5 is the largest power of 5 that is a factor of either 36 or 40. Thus

$$LCM(36,40) = 2^3 \cdot 3^2 \cdot 5 = 360$$

(b) We consider the prime factorizations of 20 and 27:

$$20 = 2^2 \cdot 5 \qquad 27 = 3^3$$

We see that no prime number is a factor of both 20 and 27. In this case, we say that

$$GCF(20,27) = 1$$

We see that 2^2 is the largest power of 2 that is a factor of either 20 or 27, that 3^3 is the largest power of 3 that is a factor of either 20 or 27, and that 5 is the largest power of 5 that is a factor of either 20 or 27. Thus

$$LCM(20,27) = 2^2 \cdot 3^3 \cdot 5 = 540.$$

(c) The prime factorizations of the two numbers are given to us in this case:

$$288 = 2^5 \cdot 3^2 \qquad 4536 = 2^3 \cdot 3^4 \cdot 7$$

We see that 2^3 is the largest power of 2 that is a factor of both $2^5 \cdot 3^2$ and $2^3 \cdot 3^4 \cdot 7$, and that 3^2 is the largest power of 3 that is a factor of both $2^5 \cdot 3^2$ and $2^3 \cdot 3^4 \cdot 7$. Thus

$$GCF(288,4536) = GCF(2^5 \cdot 3^2, 2^3 \cdot 3^4 \cdot 7) = 2^3 \cdot 3^2 = 72$$

We see that 2^5 is the largest power of 2 that is a factor of either $2^5 \cdot 3^2$ or $2^3 \cdot 3^4 \cdot 7$, that 3^4 is the largest factor of 3 that is a factor of either $2^5 \cdot 3^2$ or $2^3 \cdot 3^4 \cdot 7$, and that 7 is the largest power of 7 that is a factor of either $2^5 \cdot 3^2$ or $2^3 \cdot 3^4 \cdot 7$. Thus

$$LCM(288,4356) = LCM(2^5 \cdot 3^2, 2^3 \cdot 3^4 \cdot 7) = 2^5 \cdot 3^4 \cdot 7 = 18{,}144$$

The techniques we use for finding the greatest common factor and least common multiple for two positive integers can be extended to two or more algebraic expressions by regarding the variables involved as prime numbers.

Example 5. Find the greatest common factor and the least common multiple of the given pair of algebraic expressions.

(a) $x^7y^8z^5$, $x^4y^{12}w$ **(b)** $81x^3y^5$, $45x^8y^2$

Solution. **(a)** We treat the variables x, y, z and w as prime numbers and we consider the prime factorizations:

$$x^7y^8z^5 \qquad x^4y^{12}w$$

We see that x^4 is the largest power of x that is a factor of both $x^7y^8z^5$ and $x^4y^{12}w$, and that y^8 is the largest power of y that is a factor of both $x^7y^8z^5$ and $x^4y^{12}w$. There are no other factors of both $x^7y^8z^5$ and $x^4y^{12}w$. Thus

$$GCF(x^7y^8z^5, x^4y^{12}w) = x^4y^8$$

We see that x^7 is the largest power of x that is a factor of either $x^7y^8z^5$ or $x^4y^{12}w$, that y^{12} is the largest power of y that is a factor of either $x^7y^8z^5$ or $x^4y^{12}w$, that z^5 is the largest power of z that is a factor of either $x^7y^8z^5$ or $x^4y^{12}w$, and that w is the largest power of w that is a factor of either $x^7y^8z^5$ or $x^4y^{12}w$. Thus

$$LCM(x^7y^8z^5, x^4y^{12}w) = x^7y^{12}z^5w$$

(b) We first find the prime factorizations of $81 = 3^4$ and $45 = 3^2 \cdot 5$. We then write the algebraic expressions in factored form as

$$3^4x^3y^5 \qquad 3^2 \cdot 5x^8y^2$$

We see that 3^2 is the largest power of 3 that is a factor of both $3^4x^3y^5$ and $3^2 \cdot 5x^8y^2$, that x^3 is the largest power of x that is a factor of both $3^4x^3y^5$ and $3^2 \cdot 5x^8y^2$, and that y^2 is the largest power of y that is a factor of both $3^4x^3y^5$ and $3^2 \cdot 5x^8y^2$. Thus

$$GCF(81x^3y^5, 45x^8y^2) = GCF(3^4x^3y^5, 3^2 \cdot 5x^8y^2) = 3^2x^3y^2 = 9x^3y^2$$

We see that 3^4 is the largest power of 3 that is a factor of either $3^4x^3y^5$ or $3^2 \cdot 5x^8y^2$, that 5 is the largest factor of 5 that is a factor of either $3^4x^3y^5$ or $3^2 \cdot 5x^8y^2$, that x^8 is the largest factor of x that is a factor of either $3^4x^3y^5$ or $3^2 \cdot 5x^8y^2$, and that y^5 is the largest power of y that is a factor of either $3^4x^3y^5$ or $3^2 \cdot 5x^8y^2$. Thus

$$LCM(81x^3y^5, 45x^8y^2) = LCM(3^4x^3y^5, 3^2 \cdot 5x^8y^2) = 3^4 \cdot 5x^8y^5 = 405x^8y^5$$

The definitions of greatest common factor (*GCF*) and least common multiple (*LCM*) can be extended to three or more expressions as shown in Example 5.

Example 6. Find the greatest common factor and the least common multiple of the three algebraic expressions $18x^4y^8$, $30x^7y^2$ and $12x^5y^9z^6$.

Solution. We first find the prime factorizations of $18 = 2 \cdot 3^2$, $30 = 2 \cdot 3 \cdot 5$ and $12 = 2^2 \cdot 3$. We then write the algebraic expressions in factored form as

$$2 \cdot 3^2x^4y^8 \qquad 2 \cdot 3 \cdot 5x^7y^2 \qquad 2^2 \cdot 3x^5y^9z^6$$

We see that 2 is the largest power of 2 that is a factor all three expressions, that $3$is the largest power of 3 that is a factor all three expressions, that x^4 is the largest factor of x that is a factor all three expressions, and that y^2 is the largest power of y that is a factor all three expressions.

$$GCF(18x^4y^8, 30x^7y^2, 12x^5y^9z^6) = 2 \cdot 3x^4y^2 = 6x^4y^2$$

We see that 2^2 is the largest power of 2 in at least one of the three expressions, that 3^2 is the largest power of 3 in at least one of the three expressions, that 5 is the largest power of 5 in at least one of the three expressions, that x^7 is the largest power of x in at least one of the three expressions, that y^9 is the largest power of y in at least one of the three expressions, and that z^6 is the largest power of z in at least one of the three expressions. Thus

$$LCM(18x^4y^8, 30x^7y^2, 12x^5y^9z^6) = 2^2 \cdot 3^2 \cdot 5x^7y^9z^6 = 180x^7y^9z^6$$

Exercise Set 1.2

Find the prime factorization of the given positive integer.

1. 35

2. 63

3. 88

4. 120

5. 78

6. 84

7. 68

8. 210

9. 112

10. 675

11. 612

12. 693

Find the greatest common factor (GCF) *and the least common multiple* (LCM) *of the following pairs of numbers.*

13. $9, \quad 12$

14. $13, \quad 17$

15. $8, \quad 18$

16. $14, \quad 25$

17. $6, \quad 15$

18. $12, \quad 90$

19. $50, \quad 60$

20. $54, \quad 84$

21. $45, \quad 75$

22. $28, \quad 42$

23. $2^4 \cdot 3^2, \quad 2 \cdot 3^4$

24. $2^3 \cdot 3^2, \quad 2^2 \cdot 3 \cdot 5$

25. $3^2 \cdot 5 \cdot 7, \quad 3 \cdot 7^2 \cdot 13$

26. $5^2 \cdot 11, \quad 5 \cdot 7 \cdot 11^2$

27. $2^4 \cdot 5^2 \cdot 17, \quad 2^2 \cdot 5^3$

28. $2^4 \cdot 3^2, \quad 5^2 \cdot 13$

Find the greatest common factor (GCF) and the least common multiple (LCM) of the following pairs of expressions. Treat the variables as prime numbers.

29. $a^2b^3, \quad ab^7$

30. $x^3y^2z^5, \quad x^5y^9z^2$

31. $a^7bc^3, \quad a^2b^5$

32. $x^3y^5w, \quad xz^4w^8$

33. $6x^3, \quad 15x^7$

34. $12xy, \quad 9x^3$

35. $8a^2b^7, \quad 15a^4b^2$

36. $24y^{13}, \quad 60y^{10}$

37. $51x^2w, \quad 27w^5$

38. $9x^3, \quad 8y^3$

39. $16ab^2, \quad 52a^2b^5$

40. $48x^7y^3, \quad 28y^7zw$

Find the greatest common factor (GCF) and the least common multiple (LCM) of the following triples of expressions. Treat the variables as prime numbers.

41. $8xy^2, \quad 20x^3yw, \quad 12x^4y^3w^5$

42. $21a^3b^9, \quad 9a^7b^5, \quad 15a^5b^{12}$

43. $18b^4c^3, \quad 24c^6d^2, \quad 42bc^2d^7$

44. $8x^5, \quad 9x^{12}, \quad 6x^7$

1.3. Fractions

KYOTE Standards: *CR 2, CR 3; CA 1*

Definition 1. A ***rational number*** is the quotient of one integer, called the ***numerator***, divided by another nonzero integer, called the ***denominator***. The word ***fraction*** is commonly used to denote a rational number.

Equivalent Fractions

Fractions such as $\frac{2}{3}$, $\frac{4}{6}$ and $\frac{6}{9}$ that have the same value are called ***equivalent fractions***. Multiplying or dividing the numerator and denominator of a fraction by the same nonzero integer yields an equivalent fraction. A fraction can be reduced to an equivalent fraction in ***lowest terms*** by dividing its numerator and denominator by the greatest common factor of the numerator and denominator.

Example 1. Reduce the given fraction to lowest terms.

(a) $\frac{8}{36}$ **(b)** $\frac{36}{84}$ **(c)** $\frac{3^4 \cdot 5^3 \cdot 7}{2^3 \cdot 3 \cdot 5^5 \cdot 7^2}$

Solution. **(a)** The greatest common factor (*GCF*) of 8 and 36 is 4, and we write $8 = 4 \cdot 2$ and $36 = 4 \cdot 9$. We use these factorizations to reduce $\frac{8}{36}$ to lowest terms.

$\frac{8}{36} = \frac{4 \cdot 2}{4 \cdot 9}$ Factor out *GCF* from numerator and denominator

$= \frac{2}{9}$ Divide out (cancel) 4

Thus $\frac{8}{36}$ reduced to lowest terms is $\frac{2}{9}$.

(b) The *GCF* of $36 = 2^2 \cdot 3^2$ and $84 = 2^2 \cdot 3 \cdot 7$ is $2^2 \cdot 3 = 12$, and we write $36 = 12 \cdot 3$ and $84 = 12 \cdot 7$. We use these factorizations to reduce $\frac{36}{84}$ to lowest terms.

$\frac{36}{84} = \frac{12 \cdot 3}{12 \cdot 7}$ Factor out *GCF* from numerator and denominator

$= \frac{3}{7}$ Divide out (cancel) 12

Thus $\frac{36}{84}$ reduced to lowest terms is $\frac{3}{7}$.

(c) The fraction is given to us in factored form in this case, but we follow the same procedure as in the previous two examples. The *GCF* of $3^4 \cdot 5^3 \cdot 7$ and $2^3 \cdot 3 \cdot 5^5 \cdot 7^2$ is $3 \cdot 5^3 \cdot 7$, and we write $3^4 \cdot 5^3 \cdot 7 = (3 \cdot 5^3 \cdot 7)(3^3)$ and $2^3 \cdot 3 \cdot 5^5 \cdot 7^2 = (3 \cdot 5^3 \cdot 7)(2^3 \cdot 5^2 \cdot 7)$. We use these factorizations to reduce $\frac{3^4 \cdot 5^3 \cdot 7}{2^3 \cdot 3 \cdot 5^5 \cdot 7^2}$ to lowest terms.

$$\frac{3^4 \cdot 5^3 \cdot 7}{2^3 \cdot 3 \cdot 5^5 \cdot 7^2} = \frac{(3 \cdot 5^3 \cdot 7)(3^3)}{(3 \cdot 5^3 \cdot 7)(2^3 \cdot 5^2 \cdot 7)}$$ Factor out GCF from numerator and denominator

$$= \frac{3^3}{2^3 \cdot 5^2 \cdot 7}$$ Divide out (cancel) $3 \cdot 5^3 \cdot 7$

Note. It is easier to use the properties of exponents to reduce $\frac{3^4 \cdot 5^3 \cdot 7}{2^3 \cdot 3 \cdot 5^5 \cdot 7^2}$ directly, but our intention is to show how the *GCF* is used.

A similar technique can be used to reduce algebraic fractions as shown in the next example.

Example 2. Reduce the algebraic fraction $\frac{24x^2y^4z^7}{18x^5y^2z}$ to lowest terms.

Solution. The *GCF* of $24x^2y^4z^7$ and $18x^5y^2z$ is $6x^2y^2z$, and we write $24x^2y^4z^7 = (6x^2y^2z)(4y^2z^6)$ and $18x^5y^2z = (6x^2y^2z)(3x^3)$. We use these factorizations to reduce $\frac{24x^2y^4z^7}{18x^5y^2z}$ to lowest terms.

$$\frac{24x^2y^4z^7}{18x^5y^2z} = \frac{(6x^2y^2z)(4y^2z^6)}{(6x^2y^2z)(3x^3)}$$ Factor out GCF from numerator and denominator

$$= \frac{4y^2z^6}{3x^3}$$ Divide out (cancel) $6x^2y^2z$

Adding and Subtracting Fractions

Adding (or subtracting) two fractions with the same denominator is easy. We simply add (or subtract) the numerators of the two fractions and divide the result by the common denominator of the two fractions. When the two denominators are different, we need to rewrite the fractions as equivalent fractions with a common denominator before we can add or subtract them. Generally, we choose the least common denominator (*LCD*). The next example shows how this is done.

Example 3. Add or subtract the given fractions and reduce your answer to lowest terms.

(a) $\frac{5}{12}+\frac{1}{4}$ **(b)** $\frac{2}{3}-\frac{4}{5}$ **(c)** $\frac{7}{12}+\frac{3}{16}$ **(d)** $\frac{1}{2\cdot 3^3}+\frac{7}{2^2\cdot 3^2\cdot 5}$

***Solution*. (a)** The least common denominator (*LCD*) 12 is the least common multiple of 12 and 4. We then write $\frac{5}{12}$ and $\frac{1}{4}$ as equivalent fractions with denominator 12, add the resulting fractions and reduce to obtain

$\frac{5}{12}+\frac{1}{4}=\frac{5}{12}+\frac{1\cdot 3}{4\cdot 3}$	Write each fraction as an equivalent fraction with denominator 12
$=\frac{5}{12}+\frac{3}{12}$	Simplify
$=\frac{8}{12}$	Add
$=\frac{2}{3}$	Reduce

(b) The *LCD* is $3\cdot 5=15$. We write $\frac{2}{3}$ and $\frac{4}{5}$ as equivalent fractions with denominator 15 and subtract to obtain

$\frac{2}{3}-\frac{4}{5}=\frac{2\cdot 5}{3\cdot 5}-\frac{4\cdot 3}{5\cdot 3}$	Write each fraction as an equivalent fraction with denominator 15
$=\frac{10}{15}-\frac{12}{15}$	Simplify
$=-\frac{2}{15}$	Subtract

(c) The *LCD* of $12 = 2^2 \cdot 3$ and $16 = 2^4$ is $48 = 2^4 \cdot 3$. Thus 48 is a common multiple of 12 and 16, and we can write $48 = 12 \cdot 4$ and $48 = 16 \cdot 3$. We write $\frac{7}{12}$ and $\frac{3}{16}$ as equivalent fractions with denominator 48 and add to obtain

$\frac{7}{12} + \frac{3}{16} = \frac{7 \cdot 4}{12 \cdot 4} + \frac{3 \cdot 3}{16 \cdot 3}$	Write each fraction as an equivalent fraction with denominator 48
$= \frac{28}{48} + \frac{9}{48}$	Simplify
$= \frac{37}{48}$	Add

(d) The *LCD* of $2 \cdot 3^3$ and $2^2 \cdot 3^2 \cdot 5$ is $2^2 \cdot 3^3 \cdot 5$. Thus $2^2 \cdot 3^3 \cdot 5$ is the least common multiple of $2 \cdot 3^3$ and $2^2 \cdot 3^2 \cdot 5$, and we can write $2^2 \cdot 3^3 \cdot 5 = (2 \cdot 3^3)(2 \cdot 5)$ and $2^2 \cdot 3^3 \cdot 5 = (2^2 \cdot 3^2 \cdot 5)(3)$. We write $\frac{1}{2 \cdot 3^3}$ and $\frac{7}{2^2 \cdot 3^2 \cdot 5}$ as equivalent fractions with denominator $2^2 \cdot 3^3 \cdot 5 = 540$ and add to obtain

$\frac{1}{2 \cdot 3^3} + \frac{7}{2^2 \cdot 3^2 \cdot 5}$	Given sum of fractions
$= \frac{2 \cdot 5}{(2 \cdot 3^3)(2 \cdot 5)} + \frac{7 \cdot 3}{(2^2 \cdot 3^2 \cdot 5) \cdot 3}$	Write each fraction as an equivalent fraction with denominator $2^2 \cdot 3^3 \cdot 5$
$= \frac{10}{540} + \frac{21}{540}$	Simplify
$= \frac{31}{540}$	Add

Once we understand how to add and subtract numerical fractions, we can add and subtract algebraic fractions using the same techniques.

Example 4. Add or subtract the given algebraic fractions.

(a) $\frac{4}{x} - \frac{3}{y}$ **(b)** $\frac{5b}{6a^2} + \frac{3c}{4a}$

Solution. **(a)** The *LCD* is $x \cdot y = xy$. We write $\frac{4}{x}$ and $\frac{3}{y}$ as equivalent fractions with denominator xy and subtract to obtain

$$\frac{4}{x}-\frac{3}{y}=\frac{4\cdot y}{x\cdot y}-\frac{3\cdot x}{y\cdot x}$$

Write each fraction as an equivalent fraction with denominator xy

$$=\frac{4y-3x}{xy}$$

Subtract

(b) The *LCD* of $6a^2$ and $4a$ is $12a^2$. We write $\frac{5b}{6a^2}$ and $\frac{3c}{4a}$ as equivalent fractions with denominator $12a^2$ and add to obtain

$$\frac{5b}{6a^2}+\frac{3c}{4a}=\frac{5b\cdot 2}{6a^2\cdot 2}+\frac{3c\cdot 3a}{4a\cdot 3a}$$

Write each fraction as an equivalent fraction with denominator $12a^2$

$$=\frac{10b+9ac}{12a^2}$$

Add

Mixed Numbers

An *improper* fraction whose numerator is greater than its denominator can be written as a mixed number. For example, $\frac{23}{5}$ is written $4\frac{3}{5}$. Conversely, a mixed number such as $2\frac{3}{7}$ can be written as an improper fraction $\frac{17}{7}$.

Example 5. (a) Write the improper fraction $\frac{47}{6}$ as a mixed number.

(b) Write the mixed number $2\frac{3}{8}$ as an improper fraction.

Solution. **(a)** We divide 6 into 47 to obtain a quotient of 7 and a remainder of 5. Thus $47=6\cdot 7+5$ and dividing both sides by 6 to obtain

$$\frac{47}{6}=7+\frac{5}{6}=7\frac{5}{6}$$

(b) We write $2\frac{3}{8}$ as a sum and add the resulting fractions to obtain

$$2\frac{3}{8}=2+\frac{3}{8}=\frac{16}{8}+\frac{3}{8}=\frac{19}{8}$$

Multiplying and Dividing Fractions

Multiplying and dividing fractions is easier than adding and subtracting them because we do not have to find any common denominators. To multiply two fractions, we simply find the product of their numerators and divide by the product of their denominators. To divide one fraction by another, we take the first fraction and multiply it by the reciprocal of the second. A few examples should suffice.

Example 6. Perform the indicated calculations and simplify.

(a) $\frac{2}{3}\cdot\frac{9}{4}$ **(b)** $\frac{2}{5}\div\frac{8}{15}$ **(c)** $2\frac{3}{5}\cdot 1\frac{1}{4}$ **(d)** $\left(-\frac{1}{3}\right)^2+4$

Solution. **(a)** We calculate the product of the numerators and divide by the product of the denominators, and then reduce the resulting fraction:

$\frac{2}{3}\cdot\frac{9}{4}=\frac{2\cdot 9}{3\cdot 4}$ Multiply numerators and denominators

$=\frac{3}{2}$ Divide out (cancel) 3 and 2

(b) We multiply $\frac{2}{5}$ by the reciprocal of $\frac{8}{15}$ and reduce the resulting fraction:

$\frac{2}{5}\div\frac{8}{15}=\frac{2}{5}\cdot\frac{15}{8}$ Invert $\frac{8}{15}$ and multiply

$=\frac{3}{4}$ Divide out (cancel) 5 and 2

(c) We convert the mixed numbers to improper fractions, multiply these fractions, reduce the resulting fraction, and convert it back to a mixed number:

$2\frac{3}{5}\cdot 1\frac{1}{4}=\frac{13}{5}\cdot\frac{5}{4}$ Convert to mixed numbers to improper fractions

$=\frac{13}{4}$ Divide out (cancel) 5

$=3\frac{1}{4}$ Convert to mixed number

(d) We square $-\frac{1}{3}$ and add it to 4 to obtain

$$\left(-\frac{1}{3}\right)^2 + 4 = \frac{1}{9} + 4 \qquad \text{Square } -\frac{1}{3}$$

$$= \frac{1}{9} + \frac{36}{9} \qquad \text{Write } 4 = \frac{36}{9}$$

$$= \frac{37}{9} \qquad \text{Add}$$

Example 7. Find the value of the algebraic expression at $\frac{3}{x^2} - \frac{2}{y}$ if $x = -2$ and $y = -5$.

Solution. We make the substitutions $x = -2$ and $y = -5$, then add the resulting fractions to obtain

$$\frac{3}{x^2} - \frac{2}{y} = \frac{3}{(-2)^2} - \frac{2}{(-5)} \qquad \text{Substitute } x = -2 \text{ and } y = -5$$

$$= \frac{3}{4} + \frac{2}{5} \qquad \text{Simplify}$$

$$= \frac{3 \cdot 5}{4 \cdot 5} + \frac{2 \cdot 4}{5 \cdot 4} \qquad \text{Write each fraction as an equivalent fraction with } LCD \ 4 \cdot 5 = 20$$

$$= \frac{15}{20} + \frac{8}{20} \qquad \text{Simplify}$$

$$= \frac{23}{20} \qquad \text{Add}$$

Exercise Set 1.3

Reduce the given fraction to lowest terms.

1. $\frac{6}{15}$

2. $\frac{27}{48}$

3. $\frac{12}{18}$

4. $\frac{21}{66}$

5. $\frac{2^3 \cdot 3^2}{2^2 \cdot 3^4 \cdot 5}$

6. $\frac{2 \cdot 3^4}{2^3 \cdot 3^5 \cdot 7}$

7. $\frac{2 \cdot 3^7 \cdot 5}{2^5 \cdot 3^5 \cdot 7}$

8. $\frac{2 \cdot 3^7 \cdot 5}{2^5 \cdot 3^5 \cdot 5^2}$

9. $\frac{a^7 b^2}{a^3 b^9}$

10. $\frac{a^5 b^2 c}{a^2 b^7 c^4}$

11. $\dfrac{6xy}{21x^3}$

12. $\dfrac{24x^2y^5}{16x^3y^8z^3}$

Write the mixed number as an improper fraction.

13. $7\dfrac{3}{4}$

14. $2\dfrac{4}{7}$

15. $5\dfrac{2}{3}$

16. $6\dfrac{7}{8}$

Write the improper fraction as a mixed number.

17. $\dfrac{15}{4}$

18. $\dfrac{38}{7}$

19. $\dfrac{40}{9}$

20. $\dfrac{52}{3}$

Add (or subtract) the fractions without a calculator and express the answer in reduced form. Specify the least common denominator (LCD) in each case.

21. $\dfrac{2}{3}+\dfrac{3}{4}$

22. $\dfrac{5}{6}+\dfrac{7}{15}$

23. $\dfrac{1}{6}-\dfrac{3}{2}$

24. $\dfrac{9}{4}-\dfrac{7}{10}$

25. $\dfrac{5}{3}+\dfrac{13}{2^2\cdot 3}$

26. $\dfrac{5}{2^3\cdot 3}+\dfrac{7}{2^2\cdot 3^2}$

27. $\dfrac{1}{2^2\cdot 5}-\dfrac{1}{2\cdot 5^2}$

28. $\dfrac{5}{2\cdot 7}+\dfrac{3}{2^2}$

29. $\dfrac{3}{a}+\dfrac{2}{b}$

30. $\dfrac{3}{ab}-\dfrac{2}{b^2}$

31. $\dfrac{1}{2a^2}+\dfrac{1}{6a}$

32. $\dfrac{2}{3b}-\dfrac{5}{6b^2}$

33. $\frac{a}{2}+\frac{b}{5}$

34. $\frac{a}{18}-\frac{b}{15}$

35. $\frac{x}{7}-\frac{y}{5}$

36. $\frac{x}{12}+\frac{y}{8}$

Perform the indicated calculations.

37. $\frac{2}{5}+\frac{5}{6}$

38. $\frac{2}{5}\cdot\frac{5}{6}$

39. $\frac{7}{8}-\frac{1}{6}$

40. $\frac{7}{8}\div\frac{1}{6}$

41. $3\frac{3}{5}+1\frac{2}{5}$

42. $4\frac{2}{3}-3\frac{3}{4}$

43. $3-\frac{2}{5}$

44. $\frac{3}{4}\div\frac{9}{8}$

45. $\frac{3}{4}\cdot\frac{8}{15}$

46. $\frac{2}{3}\div\frac{7}{6}$

47. $1\frac{3}{5}\cdot 3\frac{1}{4}$

48. $2\frac{3}{7}\div 1\frac{5}{7}$

49. $\left(-\frac{2}{3}\right)^2$

50. $\left(\frac{2}{-3}\right)^2$

51. $\left(\frac{-2}{3}\right)^2$

52. $-\left(\frac{2}{3}\right)^2$

53. $3-\frac{2}{5}$

54. $1\frac{2}{7}+8$

55. $\frac{3+5}{4-16}$

56. $-4+\frac{3-6}{3+2}$

57. $\frac{-6-2(-5)}{3-10}$

58. $\frac{2}{3}-\frac{(-2)5}{8-5}$

59. $\left(-\frac{3}{4}\right)^3$

60. $\left(\frac{3}{4}\right)^3$

61. $\left(-\frac{1}{2}\right)^3 \cdot \left(\frac{2}{3}\right)^2$

62. $\frac{2+3}{2^4-3^2}$

63. $2-\left(\frac{3}{5}\right)^2$

64. $\frac{3^2+4^2}{3+4}$

Find the value of the algebraic expression at the specified values of its variables.

65. $\frac{x^2-y^2}{x-y}; \quad x=3, y=-1$

66. $\frac{2x-y}{y}; \quad x=3, y=-8$

67. $\frac{1}{x}+\frac{1}{y}; \quad x=4, y=3$

68. $\frac{1}{x}-\frac{1}{y}; \quad x=-5, y=7$

69. $\frac{x}{y^3}-y^2; \quad x=-5, y=-2$

70. $\frac{x-(2-y)}{x+y}; \quad x=-2, y=-4$

71. $\frac{6xy-y^2}{y^2}; \quad x=2, y=-2$

72. $\frac{x^2-(y-y^3)}{2x+y}; \quad x=-4, y=-1$

1.4. Decimals and Percentages

KYOTE Standards: *CR 3*

Fractions and decimals are two different ways to represent numbers. You should be able to convert a fraction to a decimal and vice-versa as described in the next two examples.

Example 1. Write each fraction or mixed number as a decimal.

(a) $\frac{5}{16}$ **(b)** $2\frac{7}{20}$ **(c)** $\frac{2}{11}$

Solution. **(a)** We divide 16 into 5 using long division or a calculator to obtain

$$\frac{5}{16} = .3125$$

(b) We could convert $2\frac{7}{20}$ to an improper fraction $\frac{47}{20}$ and then divide, or we could divide 20 into 7 and add it to 2 to obtain

$$2\frac{7}{20} = 2 + \frac{7}{20} = 2 + .35 = 2.35$$

(c) We divide 11 into 7 and discover that the result is a repeating decimal:

$$\frac{2}{11} = .181818...$$

The "..." is used to indicate that the pattern continues indefinitely.

The "place-value" structure of the number system permits us to write any decimal number as a fraction or a mixed number. For example, 37.245 can be written as

$37.245 = 37 + \frac{2}{10} + \frac{4}{100} + \frac{5}{1000}$	Write 37.245 as the sum of fractions
$37 + \frac{200}{1000} + \frac{40}{1000} + \frac{5}{1000}$	Write fractions with *LCD* 1000
$= 37 + \frac{245}{1000}$	Add
$= 37 + \frac{49}{200}$	Reduce

$$= 37\frac{49}{200} \qquad \text{Write as mixed number}$$

Example 2. Write each decimal number as a fraction in reduced form.

(a) 0.375 **(b)** 9.0032

***Solution*. (a)** We write

$$0.375 = \frac{3}{10} + \frac{7}{100} + \frac{5}{1000} \qquad \text{Write } 0.375 \text{ as the sum of fractions}$$
$$= \frac{300}{1000} + \frac{70}{1000} + \frac{5}{1000} \qquad \text{Write fractions with } LCD\ 1000$$
$$= \frac{375}{1000} \qquad \text{Add}$$
$$= \frac{3 \cdot 5^3}{2^3 \cdot 5^3} \qquad \text{Factor numerator and denominator}$$
$$= \frac{3}{8} \qquad \text{Reduce}$$

(b) We write

$$9.0032 = 9 + \frac{3}{1000} + \frac{2}{10000} \qquad \text{Write } 9.0032 \text{ as the sum of fractions}$$
$$= 9 + \frac{30}{10000} + \frac{2}{10000} \qquad \text{Write fractions with } LCD\ 10000$$
$$= 9 + \frac{32}{10000} \qquad \text{Add}$$
$$= 9 + \frac{2^5}{2^4 \cdot 5^4} \qquad \text{Factor numerator and denominator}$$
$$= 9 + \frac{2}{625} \qquad \text{Reduce}$$
$$= 9\frac{2}{625} \qquad \text{Write as mixed number}$$

We use rounding frequently when working with decimals and describe this technique in Example 3.

Example 3. Round 97.38154 to **(a)** the nearest integer **(b)** the nearest tenth **(c)** the nearest hundredth **(d)** the nearest thousandth.

***Solution*. (a)** Since 0.38154 is *less than* 0.5, we round *down* so that 97.38154 to the nearest integer is 97.

(b) Since 0.08154 is *greater than* 0.05, we round *up* so that 97.38154 to the nearest tenth is 97.4. We say that 97.38154 is 97.4 correct to *one* decimal.

(c) Since 0.00154 is *less than* 0.005, we round *down* so that 97.38154 to the nearest hundredth is 97.38. We say that 97.38154 is 97.38 correct to *two* decimals.

(d) Since 0.00054 is *greater than* 0.0005, we round up so that 97.38154 to the nearest thousandth is 97.382. We say that 97.38154 is 97.382 correct to *three* decimals.

Percentages

The ability to work with percentages is most useful because percentages occur in so many important applied problems. We examine some of the basic calculations involving percentages in Example 4.

Example 4.
(a) What is 4.5% of 8000?
(b) What is $3\frac{3}{5}\%$ of 520?
(c) The number 50 is what percent of 400?
(d) The number 17 is what percent of 93? Round your answer to the nearest tenth of a percent.
(e) The number 140 is 4 percent of what number?

***Solution.* (a)** To find 4.5% of 8000, we convert 4.5% to the decimal 0.045 and multiply by 8000 to obtain

$$0.045 \cdot 8000 = 360 \qquad \text{Thus } 4.5\% \text{ of } 8000 \text{ is } 360$$

(b) To find $3\frac{3}{5}\%$ of 520, convert $3\frac{3}{5}$ to a decimal 3.6 and then convert 3.6% to the decimal 0.036. We then multiply 0.036 by 520 to obtain

$$0.036 \cdot 520 = 18.72 \qquad \text{Thus } 3\frac{3}{5}\% \text{ of } 520 \text{ is } 18.72$$

(c) We use algebra to help us solve and *understand* this problem. We let x be the unknown percent and we translate the statement "50 is $x\%$ of 400" into an equation. We convert $x\%$ to a decimal to obtain $\frac{x}{100}$. The equation we seek is therefore

$$50 = \frac{x}{100} \cdot 400$$

We solve for x to obtain $x = 12.5$. We can verify our answer by checking that 50 is 12.5% of 400.

(d) We use algebra to help us solve and *understand* this problem. We let x be the unknown percent and we translate the statement "17 is x% of 93" into an equation. We convert x% to a decimal to obtain $\frac{x}{100}$. The equation we seek is therefore

$$17 = \frac{x}{100} \cdot 93$$

We solve for x to obtain $x = \frac{17}{93} \cdot 100$ which is approximately $x \cong 18.2796$. We round to the nearest tenth of a percent to obtain 18.3%. We can verify our answer by checking that 17 is very nearly 18.3% of 93.

(e) We use algebra to help us solve and *understand* this problem as well. We are looking for a number x such that 4 percent of x is 140. We write this statement as an equation to obtain

$$0.04x = 140$$

We solve for x to obtain $x = \frac{140}{0.04} = 3500$. We can check our answer by showing that 4% of 3500 is 140.

Exercise Set 1.4

Round each number to the nearest tenth, nearest hundredth and nearest thousandth.

1. 2.76381 **2.** 251.3517 **3.** 37.469 **4.** 0.7528

Write each decimal number as a fraction in reduced form.

5. 2.8 **6.** 0.025 **7.** 1.52 **8.** 0.65

Write each fraction or mixed number in decimal form.

9. $\frac{3}{8}$ **10.** $4\frac{5}{8}$ **11.** $\frac{5}{6}$ **12.** $7\frac{3}{5}$

Write each fraction as a percentage.

13. $\frac{1}{20}$ **14.** $\frac{3}{50}$ **15.** $1\frac{2}{5}$ **16.** $\frac{5}{16}$

Write each percentage as a decimal.

17. 5% **18.** 6.3% **19.** 0.45% **20.** 0.075%

Problems involving percentages

21. What is $2\frac{1}{2}\%$ of 16,000?

22. What is 0.04% of 24,000?

23. What is $5\frac{3}{8}\%$ of 750?

24. The number 4 is what percent of 32?

25. The number 7 is what percent of 80?

26. The number 35 is what 20 percent of what number?

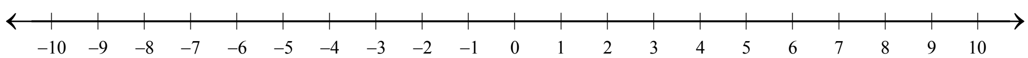
27. The number 12 is 0.80 percent of what number?

28. The number 500 is $2\frac{3}{4}$ percent of what number?

1.5. Number Line and Absolute Value

KYOTE Standards: *CR 4*

The number line is a convenient way to visualize the order properties of real numbers. The number a is *less than* the number b, written $a < b$, provided a is to the *left* of b on the number line. The number a is *greater than* the number b, written $a > b$, provided a is to the *right* of b on the number line. We see, for example, that $-5 < -2$ on the number line below since -5 is to the left of -2 and that $-2 > -5$ since -2 is to the right of -5.



We can also visualize the absolute value of a number as its distance from zero on the number line. The number 7, for example, has absolute value 7 (we write $|7| = 7$) and is 7 units to the *right* of 0 as shown on the number line above. The number -7, on the other hand, also has absolute value 7 (we write $|-7| = 7$) and is 7 units to the *left* of 0.

The absolute value of the difference of two numbers can be used to find the distance between them as shown in Example 1.

Example 1. Find the distance between the given pair of numbers by calculating the absolute value of their difference.

(a) $-9, -3$ **(b)** $-7, 6$

***Solution*. (a)** The distance 6 between -9 and -3 can be calculated by finding the absolute value of their difference $|-9-(-3)| = |-9+3| = |-6| = 6$, or equivalently, $|-3-(-9)| = |-3+9| = |6| = 6$. You can visualize this distance by plotting -9 and -3 on the number line.

(b) The distance 13 between -7 and 6 can be calculated by finding the absolute value of their difference $|-7-6| = |-13| = 13$, or equivalently, $|6-(-7)| = |6+7| = |13| = 13$. You can visualize this distance by plotting -7 and 6 on the number line.

You should be able to order real numbers on the number line whether these numbers are expressed as fractions, decimals or mixed numbers. The next example shows how two fractions can be compared without changing them to decimals.

Example 2. Determine which of the two fractions is the larger by writing them as equivalent fractions with the same least common denominator (*LCD*).

(a) $\frac{23}{36}, \frac{15}{24}$ **(b)** $-\frac{17}{12}, -1\frac{3}{8}$

Solution. **(a)** Since $36 = 2^2 \cdot 3^2$ and $24 = 2^3 \cdot 3$, the *LCD* of the two fractions is $2^3 \cdot 3^2 = 72$ and we can write the two fractions as equivalent fractions with denominator 72:

$$\frac{23}{36} = \frac{23 \cdot 2}{36 \cdot 2} = \frac{46}{72} \quad \textit{and} \quad \frac{15}{24} = \frac{15 \cdot 3}{24 \cdot 3} = \frac{45}{72}$$

We conclude that $\frac{23}{36} > \frac{15}{24}$.

(b) We first convert $-1\frac{3}{8}$ to an improper fraction $-\frac{11}{8}$ and then find the LCD of $-\frac{17}{12}$ and $-\frac{11}{8}$. The *LCD* is 24 and we can write the two fractions as equivalent fractions with denominator 24:

$$-\frac{17}{12} = -\frac{17 \cdot 2}{12 \cdot 2} = -\frac{34}{24} \quad \textit{and} \quad -\frac{11}{8} = -\frac{11 \cdot 3}{8 \cdot 3} = -\frac{33}{24}$$

We conclude that $-\frac{17}{12} < -\frac{11}{8}$ since $-\frac{34}{24}$ is to the left of $-\frac{33}{24}$ on the number line.

Example 3. Determine which of the two numbers is the smaller.

(a) $39.42178,\ 39.4231$ **(b)** $-0.87263,\ -0.872904$ **(c)** $\frac{5}{12},\ 0.416$

Solution. **(a)** The two numbers differ first in the third digit after the decimal as can be seen when they are lined up one on top of the other:

$$\begin{array}{l} 39.42178 \\ 39.4231 \end{array}$$

The third digit 1 after the decimal in the first number is less than the third digit 3 after the decimal in the second number. Thus 39.42178 is the smaller number.

(b) The two numbers differ first in the fourth digit after the decimal as can be seen when they are lined up one on top of the other:

$$\begin{array}{l} -0.87263 \\ -0.872904 \end{array}$$

The fourth digit 6 after the decimal in the first number is less than the fourth digit 9 after the decimal in the second number. Thus 0.87263 is to the left of 0.872904 on the number line, but -0.872904 is to the left of -0.87263 on the number line. Thus -0.872904 is the smaller number.

(c) We convert $\frac{5}{12}$ to the repeating decimal number $0.41666\ldots$ by dividing 12 into 5. Since $0.416 < 0.41666\ldots = \frac{5}{12}$, we see that 0.416 is the smaller number.

Example 4. Find the median of the set of numbers $\left\{\frac{7}{12}, \frac{5}{8}, \frac{1}{2}, \frac{13}{24}, \frac{2}{3}\right\}$.

Solution. We observe that 24 is the *LCD* of the five fractions and so we convert each of the fractions to an equivalent fraction with a denominator of 24. We obtain

$$\frac{7}{12} = \frac{14}{24}, \quad \frac{5}{8} = \frac{15}{24}, \quad \frac{1}{2} = \frac{12}{24}, \quad \frac{13}{24}, \quad \frac{2}{3} = \frac{16}{24}$$

It is easy to order fractions with the same denominator from smallest to largest:

$$\frac{12}{24}, \quad \frac{13}{24}, \quad \frac{14}{24}, \quad \frac{15}{24}, \quad \frac{16}{24}$$

The median is the middle number, in this case the third number, in the ordered set. Thus $\frac{14}{24} = \frac{7}{12}$ is the median.

Exercise Set 1.5

Perform the indicated calculation.

1. $|-7|$ **2.** $-|7|$ **3.** $-|-(2-5)|$ **4.** $-5-|3-8|$

Find the value of the algebraic expression at the specified values of its variable.

5. $|3-(1-x)|; \quad x=-9$

6. $\left|x^2-x^3\right|; \quad x=-2$

7. $\left|-x-x^2\right|; \quad x=-3$

8. $|-1+2(x-5)|; \quad x=3$

Place the numbers x and y on a number line and find the distance between them. Check the answer by calculating $|x-y|$.

9. $x=5, y=2$ **10.** $x=5, y=-2$ **11.** $x=-5, y=-2$ **12.** $x=-5, y=2$

13. $x=3, y=7$ **14.** $x=-4, y=-8$ **15.** $x=2, y=-7$ **16.** $x=-5, y=1$

Determine which of the two fractions is the largest (to the right of the other on the number line) by writing them as equivalent fractions with the same LCD.

17. $\frac{2}{5}, \frac{3}{8}$ **18.** $\frac{4}{9}, \frac{5}{11}$ **19.** $-\frac{5}{6}, -\frac{3}{4}$ **20.** $-\frac{7}{18}, -\frac{5}{12}$

21. $\frac{3}{15}, \frac{1}{6}$ **22.** $\frac{5}{12}, \frac{6}{14}$ **23.** $-\frac{17}{6}, -\frac{49}{18}$ **24.** $\frac{25}{4}, \frac{73}{12}$

For the number pair a, b in each exercise, write $a<b$, $a>b$, or $a=b$ depending on which of the three relationships is true.

25. $\frac{1}{3}, 0.33$ **26.** $0.24, \frac{2}{7}$ **27.** $3\frac{4}{5}, \frac{12}{5}$ **28.** $7\frac{3}{8}, \frac{59}{8}$

29. $-1\frac{5}{16}, -\frac{5}{4}$ **30.** $-\frac{15}{13}, -\frac{45}{39}$ **31.** $4.375, 4.357$ **32.** $-7.43, -7.435$

33. $-\frac{2}{3}, -0.666$ **34.** $-\frac{3}{4}, -\frac{7}{9}$ **35.** $0.0357, 0.00753$ **36.** $-0.158, -0.0581$

Find the median of the given set.

37. $\left\{\frac{1}{6}, \frac{5}{36}, \frac{1}{9}, \frac{1}{4}, \frac{5}{18}, \frac{7}{36}, \frac{2}{9}\right\}$

38. $\{-0.103, -0.132, -0.12, -0.137, -0.1, -0.117\}$

1.6. Applications Using Units, Rates and Proportions

KYOTE Standards: *CR 5*

Applications of units, rates, proportions and percentages are likely the most frequently used and most practical applications of mathematics encountered in every day life. We present a few common examples of these applications with an emphasis on how appropriate use of units can help us solve problems of this kind and achieve a deeper understanding of them.

Example 1. A large screen TV is that originally sells for \$900 is marked down to \$684. What is the percentage decrease in the price?

Solution. The price of the TV is reduced from \$900 to \$684, a reduction of \$116. The percentage decrease is found by first dividing the reduction in price by the original price $\frac{116}{900} = 0.24$. We then convert 0.24 to a percentage to obtain the percentage decrease in price of 24%.

Note. To check your work and make sense of the problem, it is a good idea to reduce 900 by 24% to obtain $900 - 0.24(900) = 684$.

Example 2. Tom, Dick and Harry are friends who go shopping. Tom buys a shirt for \$25, Dick buys a shirt for 15% more than Tom's shirt, and Harry buys a shirt for 20% more than Dick's shirt. What was the combined cost of all three shirts?

Solution. Tom paid \$25 for his shirt. Dick paid 15% more than Tom. Thus Dick paid $25 + 0.15(25)$ dollars, or $\$28.75$ for his shirt. Harry paid 20% more than Dick. Thus Harry paid $28.75 + 0.2(28.75)$ dollars, or $\$34.50$ for his shirt.

The combined cost of all three shirts is therefore $25.00 + 28.75 + 34.50$ dollars, or $\$88.25$.

Example 3. (a) A man making an annual salary of \$80,000 per year gets laid off and is forced to take a part-time job paying 60% less. What is the annual salary for his part-time job?

(b) Suppose the man is making x dollars per year instead of \$80,000. What is the annual salary for his part-time job in terms of x?

***Solution.* (a)** The man's annual salary for his part-time job is 60% less than 80,000 dollars, or $80,000 - 0.6(80,000) = 32,000$ dollars.

(b) This variation of the problem in part (a) is solved in the same way but is more abstract because the initial salary is not specified. The man's annual salary for his part-time job in this case is 60% less than x dollars, or $x - 0.6x = 0.4x$ dollars.

Example 4. A group of students take a van to a football game. They average 20 miles per gallon on the 155-mile trip. How many gallons of gas do they use?

Solution. We can make sense of this problem using the relationship among the units involved and a little algebra. We let x be the gallons of gas that are used. We use the following template that describes the relationship between gallons, miles per gallon and miles.

$$gallons \times \frac{miles}{gallon} = miles$$

Notice that the gallons cancel to give miles. Given this template, we substitute in the specified quantities to obtain

$$x \; gallons \times 20 \frac{miles}{gallon} = 155 \; miles$$

We solve this equation $20x = 155$ to obtain

$$x = \frac{155}{20} = 7.75$$

Thus there were 7.75 gallons of gas used on the trip.

Example 5. A car is going 66 feet per second. How fast is the car going in miles per hour?

Solution. The relationship among the units involved in this problem is more complicated than in Example 4. This relationship has the template:

$$\frac{feet}{\sec} \times \frac{\sec}{hour} \times \frac{miles}{feet} = \frac{miles}{hour}$$

The *seconds* unit and the *feet* units cancel to leave the units *miles* per *hour*. We are either given or can look up the fact that there are 5280 feet in a mile. We use the fact that there are 60 seconds in a minute and 60 minutes in an hour to conclude that there are $60 \times 60 = 3600$ seconds in an hour. We then fill in the numbers and do the calculations.

$$66 \frac{feet}{\sec} \times 3600 \frac{\sec}{hour} \times \frac{1}{5280} \frac{miles}{feet} = 45 \frac{miles}{hour}$$

Thus 66 feet per second and 45 miles per hour are different measurements for the same speed.

Note. If we want to know what 1 foot per second is in miles per hour, we use the same template with 1 substituted for 66 to obtain $\frac{15}{22}$, or $\frac{30}{44}$. Physics texts often give the conversion from feet per second to miles per hour as 44 feet per second equals 30 miles per hour.

Example 6. Romeo and Juliet have a lover's quarrel. Juliet bolts away due east at 5 feet per second. One minute later, Romeo stomps off in the opposite direction at 4 feet per second. How far apart are they 3 minutes after Juliet leaves?

Solution. We break down this problem into two parts by calculating the distance that Juliet travels and adding it to the distance Romeo travels. Juliet travels 3 minutes and we first convert that into seconds. Although you can likely do this conversion without using the units involved, it is instructive to do this at least once. The template is:

$$\text{sec} = \text{min} \times \frac{\text{sec}}{\text{min}}$$

Since there are 60 seconds in 1 minute, we fill in the numbers and do the calculations:

$$180\,\text{sec} = 3\,\text{min} \times 60\,\frac{\text{sec}}{\text{min}}$$

We use another template to calculate the distance (in feet) Juliet travels in 180 seconds at a rate of 5 feet per second:

$$\frac{feet}{\text{sec}} \times \text{sec} = feet$$

We fill in the numbers to obtain

$$5\,\frac{feet}{\text{sec}} \times 180\ \text{sec} = 900\ feet$$

Thus Juliet travels 900 feet.

We next consider how far Romeo travels. Since he leaves one minute *after* Juliet, he travels for 2 minutes, or 120 seconds, at the rate of 4 feet per second. We use the same approach to obtain the distance Romeo travels.

$$4\,\frac{feet}{\text{sec}} \times 120\ \text{sec} = 480\ feet$$

Thus Romeo travels 480 feet.

Therefore the two lovers are 1380 feet apart 3 minutes after Juliet leaves.

Example 7. The recommended dose of a certain drug to be given to a patient is proportional to the patient's weight. If 2 milligrams of the drug is prescribed for someone weighing 140 pounds, how many milligrams should be prescribed for someone weighing 180 pounds? Round your answer to the nearest hundredth of a milligram?

Solution. We can make sense of the problem by using a little algebra and the units involved to set up a proportion. If we let x be the milligrams of the drug to be prescribed for a person weighing 180 pounds, we obtain the proportional relationship

$$\frac{x\ mg}{180\ lb} = \frac{2\ mg}{140\ lb}$$

We solve for x to obtain $x = 180 \cdot \frac{2}{140} = 2.57143$. Thus the recommended dose for a person weighing 180 pounds is 2.57 milligrams, rounded to the nearest hundredth of a milligram.

Example 8. Jack and Jill both work part-time at a local Burger Barn.

(a) Jack makes \$2 per hour less than Jill and he works 5 hours more than Jill during one week. If Jack makes \$7 dollars per hour and works 22 hours, how much do Jack and Jill combined make (in dollars) that week?

(b) Jack makes r dollars per hour less than Jill and he works h hours more than Jill during one week. If Jack makes \$7 dollars per hour and works 22 hours, how much do Jack and Jill combined make (in dollars) that week in terms of r and h?

***Solution.* (a)** Since Jack makes \$7 an hour and works 22 hours, we conclude that Jill makes \$9 per hour and works 17 hours. We can use units, if necessary, to calculate the combined earnings of Jack and Jill for the week:

$$7\,\frac{dollars}{hour} \times 22\ hours + 9\,\frac{dollars}{hour} \times 17\ hours = 307\ dollars$$

Notice that the "hour" units cancel and yield "dollars". Thus Jack and Jill combined make \$307 in that week.

(b) This problem is similar to the one in part (a) but is more abstract. It illustrates how we transition from arithmetic to algebra. Since Jack makes \$7 an hour and works 22 hours, we conclude that Jill makes $(7 + r)$ dollars per hour and works $(22 - h)$ hours. A calculation similar to the one in part (a) yields

$$7\,\frac{dollars}{hour}\times 22\ hours+(7+r)\,\frac{dollars}{hour}\times(22-h)\ hours=\big(154+(7+r)(22-h)\big)\ dollars$$

Thus in terms of r and h, Jack and Jill combined make $154+(7+r)(22-h)$ dollars in that week.

Exercise Set 1.6

1. John bought an iPad for $480 and paid $31.20 in sales tax. What was the sales tax rate?

2. A man tips a server $3.00 on meal costing $14.50. What percentage of this cost is the tip? Round to the nearest tenth of a percent.

3. The price of a shirt is reduced from $27 to $20. What is percentage decrease in the price? Round to the nearest hundredth of a percent.

4. (a) A woman whose annual salary is $55,400 gets a 4% raise. What is her new annual salary? **(b)** If her annual salary is x dollars and she gets a 4% raise, what is her new annual salary in terms of x?

5. (a) Jennifer bought a pair of running shoes selling for $75. If the sales tax was 5.8%, what was the total cost of the purchase? **(b)** If the shoes sold for x dollars, what would the total cost of the purchase be in terms of x?

6. (a) A coat that sells for $240 is marked down 20%. What is the sales price of the coat? (**b**) If the coat sells for x dollars, what would the sales price of the coat be in terms of x?

7. (a) The Dow Jones Industrial Average drops 0.25% on one day. If the average is 12, 600 at the beginning of the day, what is this average at the end of the day? **(b)** If the average is x at the beginning of the day, what is this average at the end of the day in terms of x?

8. Mark wanted to buy a new car for $22,000. The salesman told him that with rebate and discounts he could lower the price by 25%. What is the sale price of the car? What is the final cost of the car if Kentucky sales tax of 6% is added?

9. Mark wanted to buy a new car for x dollars. The salesman told him that with rebate and discounts he could lower the price by 25%. What is the sale price of the car in terms of x? What is the final cost of the car in terms of x if Kentucky sales tax of 6% is added?

10. (a) Emily got a job at Big Bob's Storage. Her weekly salary is \$700. What is her weekly take home pay if she must pay 13% federal tax and 6% sales tax on her salary? **(b)** If her weekly salary is x dollars, what is her take home pay in terms of x?

11. (a) A woman makes 15% more than her husband. If her husband's annual salary is \$45,000, what is the combined annual income of the couple? **(b)** If her husband's annual salary is x dollars, what would the couple's combined annual income be in terms of x?

12. (a) A man invests \$10,000 in two certificates of deposit, \$6,000 in the first account earning 5% annual interest and the rest in the second account earning 4% annual interest. How much annual interest does the man earn on this investment? **(b)** If he invests x dollars in the first account and the rest in the second account, express the annual interest he earns on this investment in terms of x.

13. John has \$1400 in his savings account and withdraws $\frac{2}{7}$ of it. How much is left in his account?

14. Beth has \$5400 in her savings account and withdraws $\frac{1}{6}$ of it. The next day, she withdraws $\frac{2}{9}$ of what remains. How much is left in her account after these two transactions?

15. Diana has x dollars in her savings account and withdraws $\frac{1}{6}$ of it. The next day, she withdraws $\frac{2}{9}$ of what remains. How much is left in her account after these two transactions in terms of x?

16. David wanted to buy a new suit selling for \$400. He negotiated with the salesperson and the price was reduced by $\frac{1}{5}$. He then bought the suit and paid a sales tax that was $\frac{3}{50}$ of the sales price. How much did he pay for the suit?

17. (a) Mary got a sales job at her father's company. She was paid \$9 per hour plus a 5% commission on her sales. Last week she worked 40 hours with sales of \$5560. How much was she paid? **(b)** If she worked *h* hours with sales of *S* dollars, write an expression for the amount, A, she was paid in terms of *h* and *S*.

18. (a) Lisa's job pays $8 per hour, but if she works more than 35 hours per week she is paid $1\frac{1}{2}$ times her regular salary for the overtime hours. How much is she paid if she works hours 42 hours in one week? **(b)** If she works x overtime hours in one week, how much is she paid in terms of x?

19. (a) A plumber charges $55 an hour for his labor and his assistant charges $30. If the plumber works twice as long as his assistant on a job, and his assistant works 4 hours, how much did they charge for their labor altogether? **(b)** If the assistant works x hours, how much did they charge for their labor in terms of x?

20. (a) A girl has twice as many nickels as dimes, and 3 more quarters than dimes, in her piggy bank. If she has 8 dimes in her bank, how much money, in dollars, does she have in her bank? **(b)** If she has n dimes in her bank, how much money, in dollars, does she have in her bank in terms of n?

21. (a) Michael has only $5 bills and $20 bills in his wallet. If he has 3 more $20 bills than $5 bills, and he has 7 $20 bills, how much money, in dollars, does he have in his wallet? **(b)** If he has n $20 bills, then how much money, in dollars, does he have in his wallet in terms of n?

22. Two cars pass one another going in opposite directions along a long straight road. The westbound car is going 60 miles per hour and the eastbound car is going 66 miles per hour. How far apart are the two cars after 5 minutes?

23. Two cars are initially 10 miles apart on the same straight road and are moving towards each other, one car going 30 miles per hour and the other 36 miles per hour. How far apart are the two cars 2 minutes later?

24. Two trees are standing side-by-side in the sunlight. One is 50 feet tall and the other is 20 feet tall. If the taller tree casts an 18-foot shadow, what is the length, in feet, of the shadow of the shorter tree?

25. The Body Mass Index (BMI) for two people of the same height is proportional to their weight. **(a)** If a man 6 feet tall and weighing 180 pounds has a BMI of 24.4, what is the BMI of a man 6 feet tall and weighing 200 pounds? Round the answer to the nearest tenth. **(b)** What is the weight of a man 6 feet tall with a BMI of 30? Round the answer to the nearest pound.

26. The Body Mass Index (BMI) for two people of the same height is proportional to their weight. **(a)** If a woman 5 feet, 4 inches tall and weighing 130 pounds has a BMI of 22.3, what is the BMI of a woman 5 feet, 4 inches tall and weighing 140 pounds? **(b)** What is the weight of a woman 5 feet, 4 inches tall with a BMI of 28? Round the answer to the nearest pound.

27. The time it takes a trained runner to run a 10K is proportional to the time it takes him to run a 5K. Suppose a runner has a 10K time of 37 minutes, 15 seconds and a 5K time of 18 minutes. What is the best approximation to the time it takes his friend to run a 5K if his friend runs the 10K in 39 minutes, 15 seconds? Write the answer in minutes and seconds, rounded to the nearest second.

28. A box of Oaties contains 28 grams of cereal and 140 milligrams of sodium. How many milligrams of sodium are in 5 grams of Oaties?

29. A box of Crunchies contains 59 grams of cereal and 46 grams of carbohydrates. How many grams of carbohydrates are in 6 grams of Crunchies?

30. Northern Kentucky University President Geoffrey Mearns ran a marathon (26.2 miles) in 2 hours and 16 minutes to qualify for the Olympic trials in 1984. How fast did he run in feet per second? Round your answer to the nearest hundredth. *Hint*: 5280 feet =1 mile

31. A woman is walking at the rate of 90 yards per minute. How fast is she going in feet per second?

32. A man can run 1 mile in 6 minutes. How fast is this measured in feet per second? How fast is this measured in miles per hour? *Hint*: 5280 feet=1 mile

33. A car is going 80 kilometers per hour. How fast is this measured in miles per hour? Round the answer to the nearest tenth of a mile per hour. *Hint*: 1 kilometer $\approx$ 0.621 miles

34. A car is going 90 kilometers per hour. How fast is this measured in meters per second?

35. How many minutes does it take for a woman to walk 1200 feet if she walks at the rate of 4 feet per second?

36. How many minutes does it take a man to walk a mile if he walks at the constant rate of 4 feet per second? *Hint*: 5280 feet =1 mile

37. How many centimeters are in $\frac{5}{8}$ of a meter?

38. How many meters are in $\frac{3}{7}$ of a kilometer? Round your answer to the nearest hundredth of a meter.

39. How many centimeters are in 2 kilometers?

40. How many seconds are in 3 hour and 15 minutes?

41. How many inches are in 3 yards, 2 feet and 7 inches?

42. How many inches are in one meter? *Hint*: 1 inch $\approx$ 2.54 centimeters

43. A car gets 25 miles per gallon. How many gallons of gas are needed for the car to go 180 miles?

44. A car gets 25 miles per gallon. How many gallons of gas are needed for the car to go 300 kilometers? *Hint*: 1 kilometer $\approx$ 0.621 miles

Chapter 2. Basic Geometry

2.1. Properties of Simple Geometric Figures

KYOTE Standards: *CR 6*

Geometric Formulas

Area A *and perimeter* P *of a rectangle of length* l *and width* w

$$A = lw \qquad P = 2l + 2w$$

Area A *and circumference* C *of a circle of radius* r

$$A = \pi r^2 \qquad C = 2\pi r$$

Area A *of a triangle with base* b *and height* h

$$A = \frac{1}{2}bh$$

Example 1. (a) What is the area, in square inches, of a right triangle in the plane whose horizontal leg is 3 inches longer than twice its vertical leg and whose vertical leg has length 8 inches? **(b)** What is the area, in square inches, of a right triangle in the plane whose horizontal leg is 3 inches longer than twice its vertical leg and whose vertical leg has length n inches? Write your answer in terms of n.

***Solution*. (a)** In the case of a right triangle, we can view the base of the triangle as one its legs and the height as the other leg. We choose the horizontal leg as the base and the vertical leg as the height. The height of the triangle is then 8 inches and the base is 3 inches longer than twice the height, or 19 inches. We use the formula for the area of a triangle to obtain

$$A = \frac{1}{2}bh = \frac{1}{2}\cdot 19\cdot 8 = 76$$

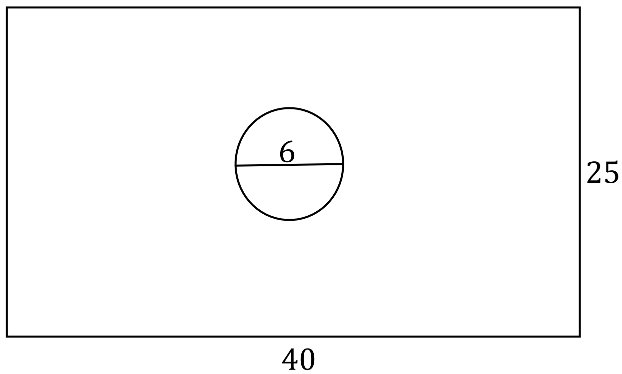
Thus the area of the triangle is 76 square inches.

(b) In this case the vertical leg has length n inches and the horizontal leg has length $2n+3$ inches. The area of the triangle is therefore

$$A = \frac{1}{2}bh = \frac{1}{2}(2n+3)n = \frac{1}{2}(2n^2+3n) = n^2 + \frac{3}{2}n$$

Thus the area of the triangle in terms of n is $n^2 + \frac{3}{2}n$ square inches.

Example 2. A planned rest area at a shopping mall will consist of a rectangular space of length 40 feet and width 25 feet, all of which will be tiled except for a circular area of diameter 6 feet as shown that will used as a green space. How much will it cost to tile the rest area if the tile selected costs \$3 per square foot? Round your answer to the nearest dollar.



Solution. The region to be tiled is inside the rectangle and outside the circle. To find its area, we calculate the area of the rectangle using the formula $A = lw$, where $l = 40$ and $w = 25$, and subtract the area of the circle using the formula $A = \pi r^2$, where $r = 3$ (half the diameter of 6), to obtain

$$Tiled\ Area = lw - \pi r^2 = 40 \cdot 25 - \pi \cdot 3^2 = 1000 - 9\pi \approx 971.726$$

We multiply this area, in square feet, by the cost of the tile, in dollars per square foot to obtain the cost of tiling the rest area.

$$Cost = (1000 - 9\pi)\, ft^2 \times 3 \frac{dollars}{ft^2} \approx 2915.18$$

The cost of tiling the rest area is therefore $\$2,915$, rounded to the nearest dollar.

Exercise Set 2.1

1. What is the area, in square centimeters, of a rectangle whose length is 120 centimeters and whose width is 80 centimeters.

2. What is the area, in square inches, of a circle whose diameter is 9 inches?

3. What is the length, in feet, of a rectangle whose width is 7 feet and whose area is 63 square feet?

4. What is the perimeter, in inches, of a rectangle whose length is 12 feet and whose area is 96 square inches?

5. What is the perimeter of a square whose area is 144 square feet?

6. One angle of a triangle has measure 26 degrees and another has measure 35 degrees. What is the measure, in degrees, of the third angle?

7. (a) What is the perimeter, in centimeters, of a rectangle whose width is 7 centimeters and whose length is 4 centimeters more than its width? **(b)** What is the perimeter, in centimeters, of a rectangle whose width is n centimeters and whose length is 4 centimeters more than its width, in terms of n?

8. (a) What is the area, in square centimeters, of a rectangle whose width is 8 centimeters and whose length is 6 centimeters more than half its width? **(b)** What is the area, in square centimeters, of a rectangle whose width is n centimeters and whose length is 6 centimeters more than half its width, in terms of n?

9. (a) What is the perimeter, in feet, of a rectangle whose width is 6 feet and whose length is 3 feet more than twice its width? **(b)** What is the perimeter, in feet, of a rectangle whose width is n feet and whose length is 3 feet more than twice its width, in terms of n?

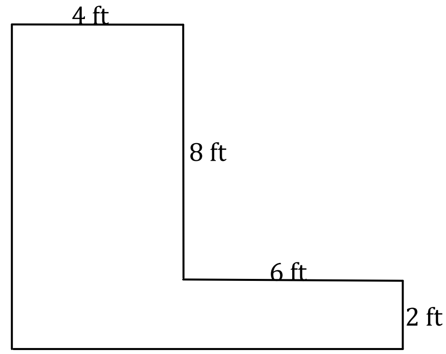
10. (a) What is the area, in square feet, of a rectangle whose width is 6 feet and whose length is 2 feet more than three times its width? **(b)** What is the area, in square feet, of a rectangle whose width is n feet and whose length is 2 feet more than three times its width, in terms of n?

11. (a) What is the area of a triangle, in square meters, whose base is 7 meters and whose height is half its base? **(b)** What is the area of a triangle, in square meters, whose base is b meters and whose height is half its base, in terms of b?

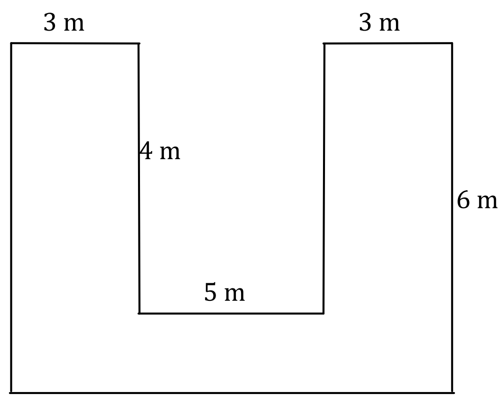
12. (a) Janis wants to carpet her living room, which measures 15 feet by 12 feet. She picked out a nice Berber style that cost $2 per square foot. How much will it cost to carpet her living room? **(b)** If the room is *L* feet by *W* feet, write an expression for the cost, C, to carpet the living room in terms of *L* and *W*.

13. (a) A circle is inscribed inside a square of side length 8 centimeters. What is the area, in square centimeters, inside the square and outside the circle? Round your answer to the nearest tenth of a square centimeter. **(b)** A circle is inscribed inside a square of side length s centimeters. What is the area, in square centimeters, inside the square and outside the circle in terms of s?

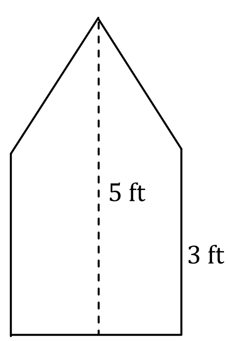
14. The figure below is a diagram of a floor that needs to be carpeted. **(a)** What is the area, in square feet, of the region that needs to be carpeted? **(b)** What is the length, in feet, of the boundary of this region? **(c)** What is the cost of carpeting this region if the carpet selected costs $2.50 per square foot?



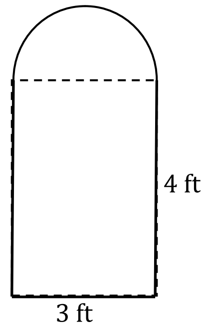
15. The figure below is a diagram of a floor that needs to be tiled. **(a)** What is the area, in square meters, of the region that needs to be tiled? **(b)** What is the length, in meters, of the boundary of this region? **(c)** What is the cost of tiling this region if the tile selected costs $12.00 per square meter?



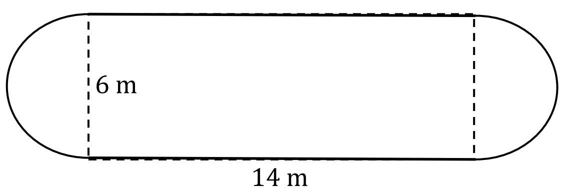
16. The metal plate shown consists of a square with a triangle on top. How much does the plate weigh, in pounds, if the metal weighs 2.5 pounds per square foot?



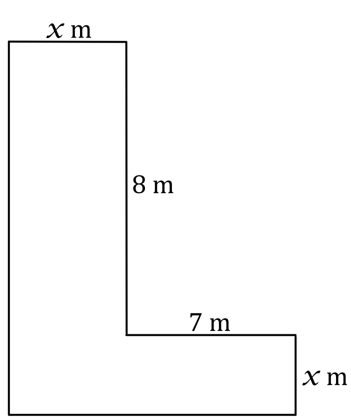
17. The figure below is a diagram of a window in the shape of a semicircle atop a rectangle of length 3 feet and height 4 feet. **(a)** What is the area, in square feet, enclosed by the window? **(b)** What is the distance, in feet, around the boundary of the window?



18. The figure below is a diagram of a garden plot in the shape of a rectangle of length 14 meters long and 6 meters wide with semicircles adjoined on each end. **(a)** A landscaping company charges $1.50 per square meter to tend the garden for each month during the summer. How much, in dollars, does the landscaping company charge each month to care for the garden? Round the answer to the nearest cent. **(b)** A fence costing $20 per meter is built around the boundary of the garden. What is the cost, in dollars, of the fencing? Round your answer to the nearest cent.



19. (a) Express the area, in square meters, enclosed by figure below in terms of x.
(b) Express the length, in meters, of the boundary of this figure in terms of x.



2.2. Coordinate Geometry

KYOTE Standards: *CR 7*

Example 1. Plot the following pairs of points in the coordinate plane and find the distance between them.

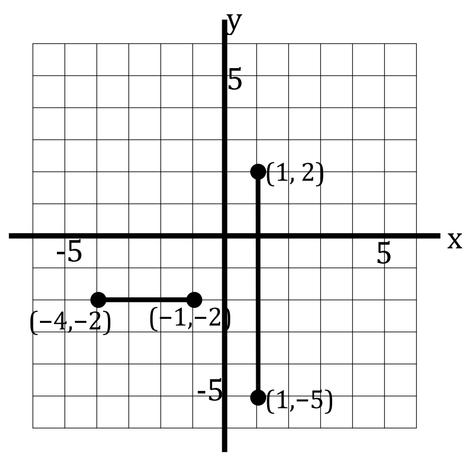
(a) $(-4,-2),(-1,-2)$ **(b)** $(1,2),(1,-5)$

***Solution*. (a)** The points $(-4,-2)$ and $(-1,-2)$, plotted on the axes below, lie along the same horizontal line. We can see from the graph that the distance between the points is 3. We could also find this distance by finding the absolute value of the difference between the x-coordinates to obtain the same result:

$$|-4-(-1)| = |-4+1| = |-3| = 3$$

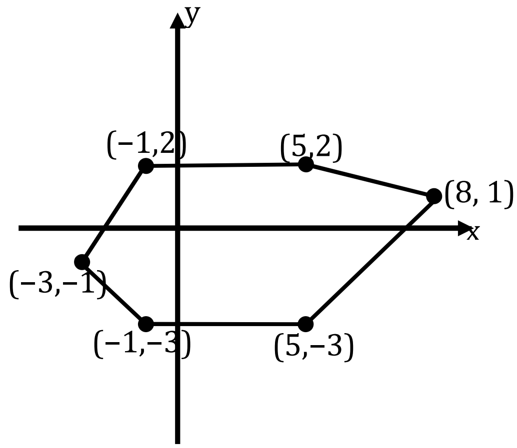
(b) The points $(1,2)$ and $(1,-5)$ plotted on the axes below lie along the same vertical line. We can see from the graph that the distance between the points is 7. We could also find this distance by finding the absolute value of the difference between the y-coordinates to obtain the same result:

$$|2-(-5)| = |2+5| = |7| = 7$$



Example 2. Sketch the polygon in the coordinate plane whose vertices are $(-1,2)$, $(5,2)$, $(8,1)$, $(5,-3)$, $(-1,-3)$, $(-3,-1)$ and back to $(-1,2)$, and whose edges are line segments connecting successive pairs of vertices. Find the area enclosed by this polygon.

Solution. The polygon is sketched below.



It can be seen from the sketch that the region enclosed by the polygon can be decomposed into a rectangle and two triangles. The vertices of the rectangle are $(-1,2)$, $(5,2)$, $(5,-3)$ and $(-1,-3)$. The rectangle has length 6 and height 5 and hence has area 30. One of the triangles has vertices $(-1,2)$, $(-1,-3)$ and $(-3,-1)$. Its base has length 5 (the length of the line segment from $(-1,2)$ to $(-1,-3)$) and its height has length 2 (the length of the line segment from $(-3,-1)$ to $(-1,-1)$). The area of this triangle is $\frac{1}{2}\cdot 5\cdot 2 = 5$. The other triangle has vertices $(5,2)$, $(8,1)$ and $(5,-3)$. Its base has length 5 and its height has length 3. The area of this triangle is $\frac{1}{2}\cdot 5\cdot 3 = \frac{15}{2}$.

The area enclosed by the polygon is $42\frac{1}{2}$, the sum of these three areas:

$$30+5+\frac{15}{2}=\frac{85}{2}=42\frac{1}{2}$$

Exercise Set 2.2

1. Plot the following pairs of points in the coordinate plane and find the distance between them in each case.

a) $(2,3), (7,3)$ **b)** $(-2,5), (-2,-3)$ **c)** $(-8,5), (-2,5)$

d) $(-9,1), (6,1)$ **e)** $(0,-4), (0,2)$ **f)** $(-5,0), (-1,0)$

g) $\left(\frac{2}{5},2\right), \left(\frac{8}{3},2\right)$ **h)** $\left(-\frac{2}{3},-1\right), \left(\frac{7}{4},-1\right)$ **i)** $\left(3,\frac{5}{6}\right), \left(3,-\frac{15}{8}\right)$

2. Sketch the rectangle in the coordinate plane whose vertices are $(-2,5)$, $(3,5)$, $(3,1)$ and $(-2,1)$. Find its area and its perimeter.

3. Sketch the rectangle in the coordinate plane whose vertices are $(-7,-2)$, $(-3,-2)$, $(-3,-5)$ and $(-7,-5)$. Find its area and its perimeter.

4. Sketch the circle in the coordinate plane given that the line segment connecting $(0,2)$ and $(6,2)$ is a diameter of the circle. Find two additional points that lie on the circle. Find the area and the circumference of the circle.

5. Sketch the circle in the coordinate plane given that the line segment connecting $(8,1)$ and $(3,1)$ is a diameter of the circle. Find two additional points that lie on the circle. Find the area and the circumference of the circle.

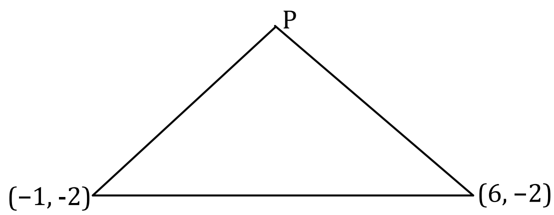
6. Sketch the triangle in the coordinate plane whose vertices are $(-6,0)$, $(-1,0)$ and $(-3,-4)$. Find its area.

7. Sketch the triangle in the coordinate plane whose vertices are $(5,4)$, $(-2,4)$ and $(-1,2)$. Find its area.

8. Sketch the polygon in the coordinate plane whose vertices are $(-2,5)$, $(4,5)$, $(4,-2)$, $(1,-6)$, $(-2,-2)$ and back to $(-2,5)$, and whose edges are line segments connecting successive pairs of vertices. Find the area enclosed by this polygon.

9. (a) What are the coordinates of the vertices P and Q of the rectangle below if its area is 35? **(b)** What are the coordinates of the vertices P and Q of the rectangle below if its perimeter is 22?

10. What is the y-coordinate of the vertex P of the triangle below if its area is 14?



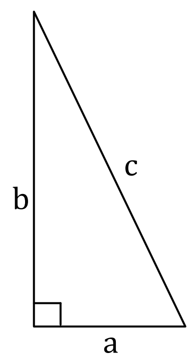
2.3. Pythagorean Theorem and Similar Triangles

KYOTE Standards: *CA 15*

Pythagorean Theorem

The Pythagorean theorem is the best-known and most frequently used theorem in high school and college mathematics. It provides a simple relationship between the two legs of a right triangle and its hypotenuse. Recall that a right triangle has one 90^o angle called a right angle, that its hypotenuse is the side opposite the right angle and that the legs are the other two sides of the triangle.

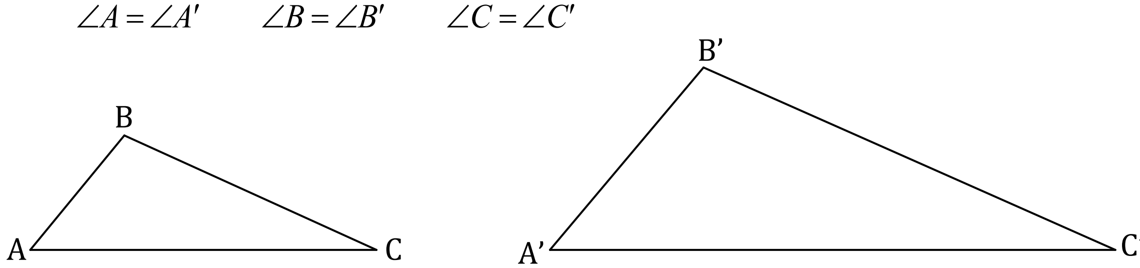
The theorem states that the sum of the squares of the legs of a right triangle equals the square of its hypotenuse. For the triangle below, this result can be stated in symbols as $a^2 + b^2 = c^2$.



Similar Triangles

Two triangles are similar if they have the same *shape* but not necessarily the same *size*. This means that corresponding angles of two similar triangles are equal. In the two similar triangles ABC and $A'B'C'$ shown below we have

$$\angle A = \angle A' \qquad \angle B = \angle B' \qquad \angle C = \angle C'$$



The most important feature of similar triangles is that corresponding sides of two similar triangles are *proportional*. For the two similar triangles ABC and $A'B'C'$

above, this means that there is a positive number k (called the *constant of proportionality*) such that

$$|AB| = k|A'B'| \qquad |AC| = k|A'C'| \qquad |BC| = k|B'C'|$$

Here we use the notation $|AB|$ to denote the length of the line segment AB. Triangle ABC is *smaller* than triangle $A'B'C'$ if $k < 1$ as in our example; triangle ABC is *larger* than triangle $A'B'C'$ if $k > 1$; and triangle ABC is the same size as triangle $A'B'C'$ if $k = 1$. We call two similar triangles with the same size *congruent*.

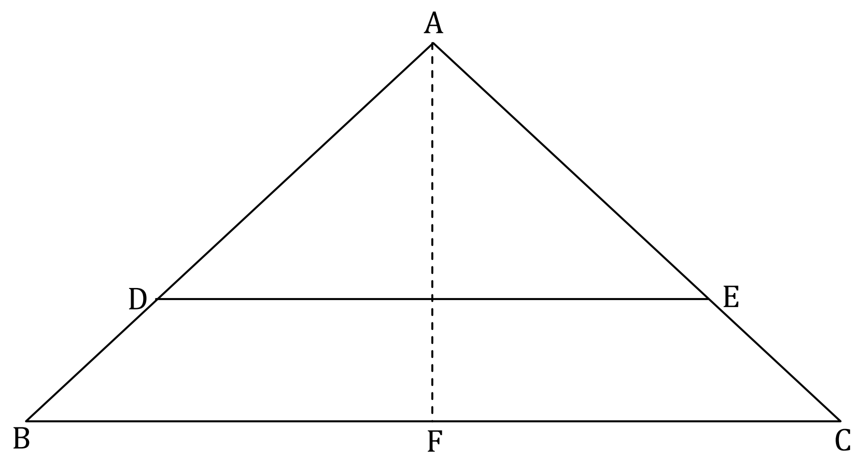
If we solve each of the three equations above for k and set them equal, we obtain

$$k = \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}$$

These ratios are used to solve many problems in plane geometry.

Example 1. In the figure shown, the isosceles triangle ABC represents the cross-section of a roof with a reinforcing strut DE. The base BC of the roof has length 20 feet. The slant portion of the roof AD has length 10 feet, and the slant portion of the roof DB has length 4 feet.

- **(a)** Find the length of the strut DE, in feet. Round the answer to the nearest tenth of a foot.
- **(b)** Find the height (the length of the line segment AF) of the roof, in feet. Round the answer to the nearest tenth of a foot.



Solution. **(a)** The triangles ABC and ADE are similar and thus the lengths of their sides are proportional. We set up the proportional relationship as

$$\frac{|DE|}{|BC|} = \frac{|AD|}{|AB|}$$

We use the notation $|DE|$ to denote the length of DE and the same notation for the other line segments. We then have

$$\frac{|DE|}{20} = \frac{10}{14}$$

We solve for $|DE|$ to obtain

$$|DE| = 20 \cdot \frac{10}{14} \approx 14.2857$$

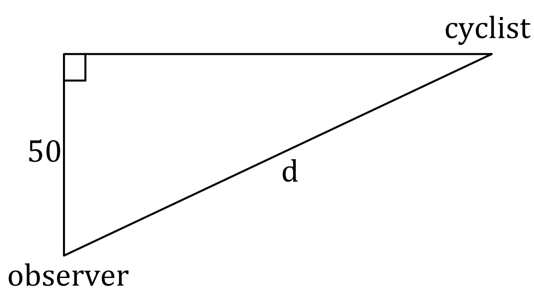
The length of the strut is therefore 14.3 when rounded to the nearest tenth of a foot.

(b) The line segment AF is the perpendicular bisector of the base BC of the triangle ABC. Thus the angle $\angle AFB$ is a right angle and the length of the line segment BF is 10 feet. We use the Pythagorean theorem to find the unknown length $|AF|$ of the right triangle AFB. We obtain

$$14^2 = |AF|^2 + 10^2 \quad or \quad |AF|^2 = 14^2 - 10^2 = 96$$

We take the square root to obtain $|AF| = \sqrt{96} \approx 9.79796$. Thus the height of the roof is 9.8 feet when rounded to the nearest tenth of a foot.

Example 2. An observer spots a cyclist 50 feet north of him going east at 18 feet per second as shown in the diagram. How far apart are the observer and the cyclist 5 seconds after the observer spots the cyclist? Round the answer to the nearest foot.



Solution. We first calculate the distance traveled by the cyclist 5 seconds after she is spotted by the observer. We obtain

$$18\frac{feet}{\sec} \times 5\sec = 90\ feet$$

We use the Pythagorean theorem to find the distance d between the observer and the cyclist 5 seconds after he spots her. We obtain

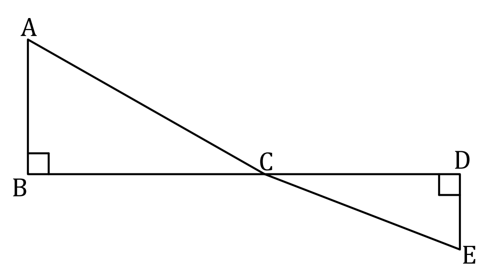
$$d^2 = 50^2 + 90^2 = 10600$$

We take the square root to obtain $d = \sqrt{10600} \approx 102.956$. Thus the distance between the observer and the cyclist 5 seconds after he spots her is 103 feet when rounded to the nearest foot.

Exercise Set 3.3

1. A woman 5 feet tall stands next to her daughter who is 4 feet tall on a sunny day. If the woman's shadow has length 3 feet, what is the length, in feet, of her daughter's shadow?

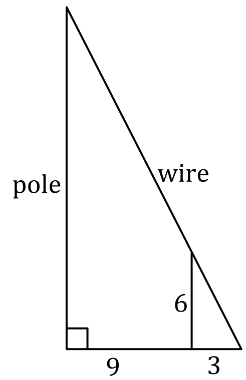
2. In the figure below, AB has length 6 feet, BC has length 8 feet and CD has length 7 feet. If angles A and D are right angles, what is the length of DE rounded to the nearest tenth of a foot?



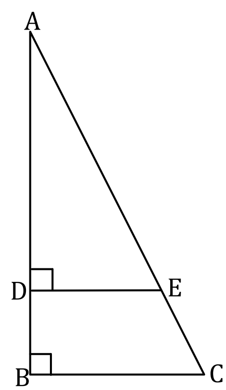
3. If the two sides of a right triangle both have length 5 inches, what is the length, in inches, of the hypotenuse?

4. If the hypotenuse of a right triangle has length 9 centimeters and one of its sides has length 5 centimeters, what is the length of its other side, in centimeters?

5. In the figure below, a wire is stretched from the top of a pole to the ground. A man 6 feet tall stands 9 feet from the base of the pole and 3 feet from where the wire is attached to the ground so that his head touches the wire. What is the height of the pole?



6. In the triangle shown below, AD has length 7 inches, AB has length 9 inches and DE has length 4 inches. If angles B and D are right angles, what is the area of triangle ABC rounded to the nearest tenth of a square inch?

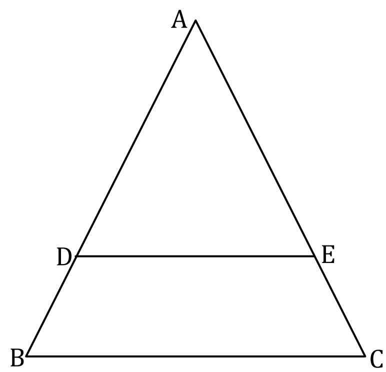


7. Two cyclists leave an intersection together. One goes north at 15 feet per second and the other goes east at 9 feet per second. How far apart are the two cyclists after one minute? Round your answer to the nearest foot.

8. An equilateral triangle has side length 10 centimeters. Find its area in square centimeters. *Hint*: The altitude of an equilateral triangle is the perpendicular bisector of its base.

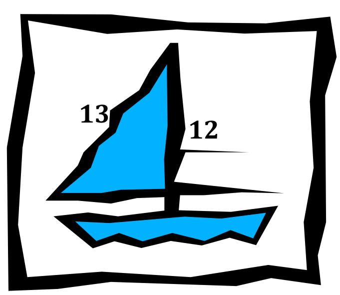
9. A plywood surface is in the shape of an isosceles triangle with two sides of length 9 feet and base of length 6 feet. How much will this plywood surface cost if plywood costs $12 per square foot? Round your answer to the nearest cent. *Hint*: The altitude of an isosceles triangle is the perpendicular bisector of its base.

10. In the figure below, ABC is an isosceles triangle, DE is parallel to BC, AD has length 10 inches, AB has length 12 inches and DE has length 8 inches. What is the area of triangle ABC to the nearest square inch?

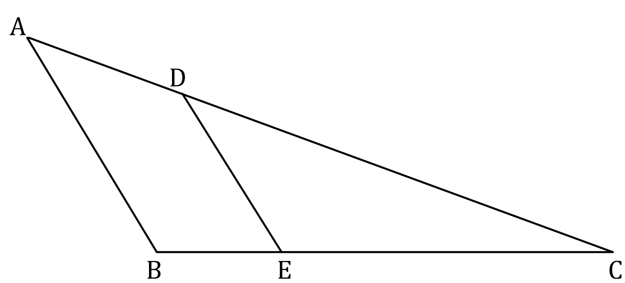


11. A truck crosses an intersection going north at 40 miles per hour. One half hour later, a car crosses the same intersection going east at 50 miles per hour. How many miles apart are the car and the truck 15 minutes after the car leaves the intersection?

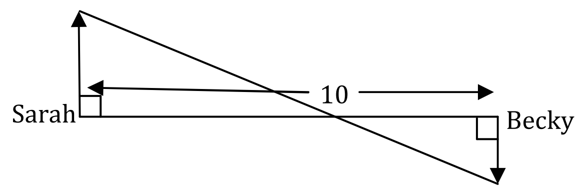
12. The sail on the sailboat shown below forms a right triangle whose hypotenuse has length 13 feet and whose vertical side has length 12 feet. If cloth for the sail costs $10 per square foot, what is the cost of the sail?



13. In the figure below, DE is parallel to AB, AB has length 8 centimeters, BE has length 5 centimeters, and EC has length 7 centimeters. What is the length of DE rounded to the nearest tenth of a foot?



14. Becky is 10 miles east of Sarah when both begin running as shown in the figure below. Sarah runs north at 8 miles per hour and Becky runs south at 7 miles per hour. How far apart are the two girls 30 minutes later?



Chapter 3. Exponents

3.1. Integer Exponents

KYOTE Standards: *CR 10; CA 1, CA 3*

We looked at some arithmetic problems involving exponents in Chapter 1. We introduced exponential notation, observing that the symbol $5^3 = 5 \cdot 5 \cdot 5$ represents the product of 3 factors of 5 multiplied together. We generalize this statement in Definition 1.

Definition 1. If a is a real number and n is a positive integer, then a to the power n is

$$a^n = a \cdot a \cdot \ldots \cdot a \qquad (n \textit{ factors of } a)$$

The number a is called the *base* and the number n is called the *exponent*.

We did not discuss properties of exponents in Chapter 1, but we could have discovered them. For example, notice that

$$5^3 \cdot 5^4 = (5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5 \cdot 5) = 5^7$$

We generalize this result for any base a and positive integer exponents m and n to discover the property $a^m \cdot a^n = a^{m+n}$. In words, if we multiply m factors of a by n factors of a we obtain $m+n$ factors of a.

We would like this property to be true even when m and n are 0 or negative integers. For example, if this property is true, then we must have, for any $a \neq 0$,

$$a^0 \cdot a^3 = a^{0+3} = a^3$$

If we divide by a^3, we obtain $a^0 = 1$.

Similarly, if n is a positive integer and this property is true, then we must have, for any $a \neq 0$,

$$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1$$

If we divide by a^n, we obtain $a^{-n} = \frac{1}{a^n}$. If we divide by a^{-n}, we obtain $a^n = \frac{1}{a^{-n}}$.

These observations lead to the following definition.

Definition 2. If $a \neq 0$ is any real number and n is a positive integer, then

$$a^0 = 1 \qquad and \qquad a^{-n} = \frac{1}{a^n} \qquad and \qquad a^n = \frac{1}{a^{-n}}$$

With this definition, we are able to state five properties of exponents that apply whenever the base is any nonzero real number and the exponent is any integer, positive, negative or zero. Some examples of how each property is applied are given.

Properties of Exponents

If $a \neq 0$ and $b \neq 0$ are real numbers, m and n are integers, then

Property	**Examples**	
1. $a^m a^n = a^{m+n}$	$5^3 \cdot 5^4 = 5^{3+4} = 5^7$	$5^2 \cdot 5^{-6} = 5^{2-6} = 5^{-4} = \frac{1}{5^4}$
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^7}{3^2} = 3^{7-2} = 3^5$	$\frac{3^{-8}}{3^{-2}} = 3^{-8-(-2)} = 3^{-6} = \frac{1}{3^6}$
3. $(a^m)^n = a^{mn}$	$(7^2)^3 = 7^{2\cdot 3} = 7^6$	$(7^3)^{-2} = 7^{3\cdot(-2)} = 7^{-6} = \frac{1}{7^6}$
4. $(ab)^n = a^n b^n$	$(3 \cdot 5)^2 = 3^2 \cdot 5^2$	$(3 \cdot 5)^{-2} = 3^{-2} \cdot 5^{-2} = \frac{1}{3^2} \cdot \frac{1}{5^2} = \frac{1}{3^2 \cdot 5^2}$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{5}{3}\right)^4 = \frac{5^4}{3^4}$	$\left(\frac{5}{3}\right)^{-4} = \frac{5^{-4}}{3^{-4}} = \frac{1/5^4}{1/3^4} = \frac{1}{5^4} \cdot \frac{3^4}{1} = \frac{3^4}{5^4}$

Positive Integer Exponents

We can use the properties of exponents to simplify algebraic expressions involving positive exponents as discussed in Example 1.

Example 1. Simplify the given expression, writing it without any negative exponents.

(a) $\frac{x^3}{x^5}$ **(b)** $(5u^3v^4)^2$ **(c)** $(3a^2b)(-2a^3b^4)^3$ **(d)** $\left(\frac{x^3}{4y}\right)^2\left(\frac{x^4y^2}{z}\right)^3$

Solution. **(a)** We can apply Property 2 and Definition 2 to obtain

$$\frac{x^3}{x^5} = x^{3-5} = x^{-2} \qquad \text{Property 2: } \frac{a^m}{a^n} = a^{m-n}$$

$$= \frac{1}{x^2} \qquad \text{Definition 2: } a^{-n} = \frac{1}{a^n}$$

In this example, it is easier to use Property 1 and cancel to obtain

$$\frac{x^3}{x^5} = \frac{x^3}{x^3x^2} \qquad \text{Property 1: } a^m a^n = a^{m+n}$$

$$= \frac{1}{x^2} \qquad \text{Divide out (cancel) } x^3$$

(b) We use Property 4 and Property 3 to obtain

$$(5u^3v^4)^2 = 5^2(u^3)^2(v^4)^2 \qquad \text{Property 4: } (ab)^n = a^n b^n$$

$$= 25u^6v^8 \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

(c) We use Property 4, Property 3 and Property 1 to obtain

$$(3a^2b)(-2a^3b^4)^3 = (3a^2b)(-2)^3(a^3)^3(b^4)^3 \qquad \text{Property 4: } (ab)^n = a^n b^n$$

$$= (3a^2b)(-8)a^9b^{12} \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

$$= -24a^{11}b^{13} \qquad \text{Property 1: } a^m a^n = a^{m+n}$$

(d) We use all five properties to obtain

$$\left(\frac{x^3}{4y}\right)^2\left(\frac{x^4y^2}{z}\right)^3 = \frac{(x^3)^2}{(4y)^2}\cdot\frac{(x^4y^2)^3}{z^3} \qquad \text{Property 5: } \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$= \frac{x^6}{4^2y^2}\cdot\frac{(x^4y^2)^3}{z^3} \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

$$= \frac{x^6}{4^2y^2}\cdot\frac{(x^4)^3(y^2)^3}{z^3} \qquad \text{Property 4: } (ab)^n = a^n b^n$$

$$= \frac{x^6x^{12}y^6}{16y^2z^3} \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

$$= \frac{x^{18}y^6}{16y^2z^3} \qquad \text{Property 1: } a^m a^n = a^{m+n}$$

$$= \frac{x^{18}y^4}{16z^3} \qquad \text{Property 2: } \frac{a^m}{a^n} = a^{m-n}$$

Negative Integer Exponents

We can also use the properties of exponents to simplify algebraic expressions that involve both positive and negative integer exponents. There is a strategy that can be used to simplify this process and help reduce mistakes. It involves converting an expression involving negative exponents to equivalent expression involving only positive exponents. We explain this strategy in the following example.

Example 2. Convert the expression $\dfrac{x^{-3}y^{-2}}{x^5y^{-7}z^{-4}}$ to an equivalent expression involving only positive exponents.

Solution. We write the expression as the product of single factors, apply Definition 2, and rewrite as a single expression with all positive exponents. We obtain

$$\frac{x^{-3}y^{-2}}{x^5y^{-7}z^{-4}} = x^{-3}\cdot\frac{1}{x^5}\cdot y^{-2}\cdot\frac{1}{y^{-7}}\cdot\frac{1}{z^{-4}} \qquad \text{Multiplication of fractions}$$

$$= \frac{1}{x^3}\cdot\frac{1}{x^5}\cdot\frac{1}{y^2}\cdot y^7\cdot z^4 \qquad \text{Definition 2: } a^{-n}=\frac{1}{a^n};\ a^n=\frac{1}{a^{-n}}$$

$$= \frac{y^7z^4}{x^3x^5y^2} \qquad \text{Multiplication of fractions}$$

We do not include this detailed explanation in the examples that follow. Instead, we simply write

$$\frac{x^{-3}y^{-2}}{x^5y^{-7}z^{-4}} = \frac{y^7z^4}{x^3x^5y^2} \qquad \text{Convert to positive exponents}$$

Example 3. Simplify the given expression, writing it without any negative exponents.

(a) $(-3x^2y^{-3})^2(x^{-3}y^2)^{-5}$ **(b)** $\left(\dfrac{a^2b}{c^3}\right)^{-2}$ **(c)** $\left(\dfrac{5x^{-4}y^3}{2x^2y^{-1}}\right)^{-3}$

***Solution*. (a)** We use Property 4, Property 3, Property 1 and convert to positive exponents to obtain

$$(-3x^2y^{-3})^2(x^{-3}y^2)^{-5} = (-3)^2(x^2)^2(y^{-3})^2(x^{-3})^{-5}(y^2)^{-5} \qquad \text{Property 4: } (ab)^n = a^nb^n$$

$$= 9x^4y^{-6}x^{15}y^{-10} \qquad \text{Property 3: } (a^m)^n = a^{mn}$$

$$= 9x^{19}y^{-16} \qquad \text{Property 1: } a^ma^n = a^{m+n}$$

$$= \frac{9x^{19}}{y^{16}} \qquad \text{Convert to positive exponent}$$

(b) We use Property 5, Property 4, and convert to positive exponents to obtain

$$\left(\frac{a^2b}{c^3}\right)^{-2} = \frac{(a^2b)^{-2}}{(c^3)^{-2}}$$ Property 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$$= \frac{a^{-4}b^{-2}}{c^{-6}}$$ Property 4: $(ab)^n = a^n b^n$

$$= \frac{c^6}{a^4b^2}$$ Convert to positive exponents

(c) We use Property 5, Property 4, convert to positive exponents, and then use Property 1 to obtain

$$\left(\frac{5x^{-4}y^3}{2x^2y^{-1}}\right)^{-3} = \frac{(5x^{-4}y^3)^{-3}}{(2x^2y^{-1})^{-3}}$$ Property 5: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$$= \frac{5^{-3}x^{12}y^{-9}}{2^{-3}x^{-6}y^3}$$ Property 4: $(ab)^n = a^n b^n$

$$= \frac{2^3x^{12}x^6}{5^3y^3y^9}$$ Convert to positive exponents

$$= \frac{8x^{18}}{125y^{12}}$$ Property 1: $a^m a^n = a^{m+n}$

Example 4. Find the value of the algebraic expression $\frac{x^{-3}y}{x^{-2}y^{-2}}$ when $x = -3$ and $y = -2$.

Solution. It is a good idea to simplify this complicated expression before evaluating it. We first convert to all positive exponents, simplify using Property 1 and canceling to obtain

$$\frac{x^{-3}y}{x^{-2}y^{-2}} = \frac{x^2yy^2}{x^3}$$ Convert to positive exponents

$$= \frac{x^2y^3}{x^3}$$ Property 1: $a^m a^n = a^{m+n}$

$$= \frac{x^2y^3}{x^2x}$$ Write $x^3 = x^2x$

$$= \frac{y^3}{x}$$ Divide out (cancel) x^2

It is now a simple matter to find the value of the simplified expression $\frac{y^3}{x}$ at $x = -3$ and $y = -2$ to obtain

$$\frac{(-2)^3}{-3} = \frac{-8}{-3} = \frac{8}{3}$$

Exercise Set 3.1

Simplify the expression, writing it without any negative exponents.

1. $(-2x^4)^3$

2. $(6y)^3$

3. $(-3a^4bc^5)^2$

4. $-3(-2x^2)^3$

5. $(-7a^4)^2$

6. $a^2(a^4b^3c)^5$

7. $\frac{(xy^2z^3)^4}{(x^3y^2z)^3}$

8. $\frac{(2y^3)^4}{2y^5}$

9. $\frac{(2a^3)^2(3a^4)}{(a^3)^4}$

10. $\left((-x^3y)^2z^4\right)^3$

11. $\left(\frac{-u^2v^3}{4u^4v}\right)^2$

12. $\left(\frac{2xy^6z}{xy^2z^3}\right)^3$

13. $(2x^3)^{-3}$

14. $(ab^2)^{-7}$

15. u^3u^{-9}

16. $(-2x^5)^{-2}$

17. $(x^2y^{-3})^{-4}$

18. $(-u^{-4})^3(2u^5)^{-2}$

19. $(-4x^2y^{-7})^3$

20. $\left(-3x^{-2}y\right)^2(2x^3y^{-4})^{-2}$

21. $(-2a^3b^{-4}c^{-7})^3$

22. $\frac{a^{-3}b^4}{a^{-5}b^5}$

23. $\left(\frac{a^{-1}}{5b^4}\right)^2$

24. $\left(\frac{-2a^4}{b^2}\right)^{-3}$

25. $\left(\frac{9y}{y^{-5}}\right)^{-1}$

26. $\left(\frac{x^6}{4x^2}\right)^{-2}$

27. $\left(\frac{x^8}{x^{-2}}\right)^3\left(\frac{x^{-3}}{2x}\right)^2$

28. $\frac{(-a^2b^3)^{-2}}{a^{-4}b^2}$

29. $\left(\frac{x^{-1}yz^{-2}}{x^{-8}y^{-5}z}\right)^{-1}$

30. $\left(\frac{xy^{-2}z^{-3}}{x^2y^3z^{-4}}\right)^{-3}$

31. $(3ab^2c)\left(\frac{2a^2b}{c^3}\right)^{-2}$

32. $\frac{1}{4}x^2y^{-5}\left(3\frac{x^{-4}}{y^{-2}}\right)^2$

33. $\left(\frac{x^5}{2y^{-3}}\right)^3\left(\frac{6x^{-8}}{y}\right)^2$

Find the value of the algebraic expression at the specified values of its variable or variables without using a calculator. Simplify the expression before evaluating. Check your answer using a calculator.

34. $x^3x^{-4}; \quad x=5$

35. $(x^{-2})^3; \quad x=-2$

36. $(2x^{-1})^3; \quad x=3$

37. $(x^2+y^2)^{-1}; \quad x=-1, y=2$

38. $x^{-2}+y^{-2}; \quad x=1, y=2$

39. $(x+y)^{-1}; \quad x=3, y=5$

40. $x^{-1}+y^{-1}; \quad x=3, y=5$

41. $\left(\frac{x}{y}\right)^{-2}; \quad x=-2, y=3$

42. $\left(\frac{x}{y}\right)^{-2}; \quad x=1, y=3$

43. $\frac{x^2}{x^{-3}}; \quad x=-2$

44. $\frac{x^{-3}}{x^{-2}}; \quad x=7$

45. $\left(\frac{x^{-2}}{y}\right)^{-1}; \quad x=3, y=7$

3.2. Square Roots

KYOTE Standards: *CR 10; CA 1, CA 4*

Definition 1. The *square root* of a nonnegative number a is that unique nonnegative number, denoted by $\sqrt{a}$, whose square is a. In other words, $\sqrt{a}$ is that nonnegative number such that $(\sqrt{a})^2 = a$. The symbol $\sqrt{}$ is called a *radical sign*.

The squares of positive integers, called *perfect squares*, all have integer square roots. The number 9 is a perfect square since $9 = 3^2$, but 8 is not a perfect square since there is no integer n such that $8 = n^2$. Perfect squares play an important role in simplifying both square roots of positive integers and algebraic expressions involving positive integers.

There are two rules we use to simplify square roots of positive integers and algebraic expressions involving positive integers.

Product Rule for Square Roots

If a and b are positive real numbers, then $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Quotient Rule for Square Roots

If a and b are a positive real numbers, then $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

Example 1. Simplify the expression.

(a) $\sqrt{49}$ **(b)** $\sqrt{48}$ **(c)** $\sqrt{\dfrac{8}{25}}$

***Solution*. (a)** Since $49 = 7^2$ is a perfect square, $\sqrt{49} = 7$

(b) Since 48 is not a perfect square, we look for a factor of 48 that is a perfect square. Successive factoring gives

$$48 = 4 \cdot 12 = 4 \cdot 4 \cdot 3 = 16 \cdot 3$$

We then use the product rule for square roots to obtain

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$$

(c) We use the quotient rule and then the product rule to obtain

$$\sqrt{\frac{8}{25}} = \frac{\sqrt{8}}{\sqrt{25}} = \frac{\sqrt{4 \cdot 2}}{5} = \frac{\sqrt{4}\sqrt{2}}{5} = \frac{2\sqrt{2}}{5}$$

We can use the product and quotient rules for square roots to simplify algebraic expressions. The idea is to treat the variables involved in the same way as we treat positive integers. For example, if x is a positive real number, then $\sqrt{x^2} = x$ since x is the unique positive number that when squared is x^2. Similarly, we have $\sqrt{x^6} = \sqrt{(x^3)^2} = x^3$ and $\sqrt{x^{18}} = \sqrt{(x^9)^2} = x^9$. If the power of x is an odd integer, then we use the product rule for square roots. For example, a simplified form of $\sqrt{x^9}$ is

$$\sqrt{x^9} = \sqrt{x^8 x} = \sqrt{x^8}\sqrt{x} = x^4\sqrt{x}$$

Example 2. Simplify the expression. Assume that all variables represent positive numbers.

(a) $\sqrt{36x^4y^7}$ **(b)** $\sqrt{27x^5}$ **(c)** $\sqrt{\dfrac{8x^2y^4}{x^6y}}$ **(d)** $\sqrt{\dfrac{40x^3y^2z}{2x^{-5}y^{-3}}}$

***Solution*. (a)** We simplify the expression $\sqrt{36x^4y^7}$ as follows:

$$\sqrt{36x^4y^7} = \sqrt{36}\sqrt{x^4}\sqrt{y^7}$$ Product Rule for Square Roots

$$= 6\sqrt{x^4}\sqrt{y^6y}$$ $\sqrt{36} = 6$; Factor: $y^7 = y^6y$

$$= 6\sqrt{x^4}\sqrt{y^6}\sqrt{y}$$ Product Rule for Square Roots

$$= 6x^2y^3\sqrt{y}$$ Simplify: $\sqrt{x^4} = x^2$, $\sqrt{y^6} = y^3$

(b) We simplify the expression $\sqrt{36x^4y^7}$ as follows:

$$\sqrt{27x^5} = \sqrt{9\cdot 3x^4x}$$ Factor: $27 = 9\cdot 3$, $x^5 = x^4x$

$$= \sqrt{9}\sqrt{x^4}\sqrt{3x}$$ Product Rule for Square Roots

$$= 3x^2\sqrt{3x}$$ Simplify: $\sqrt{9} = 3$, $\sqrt{x^4} = x^2$

(c) We first simplify the expression under the radical sign of $\sqrt{\dfrac{8x^2y^4}{x^6y}}$ and then simplify the resulting expression. We obtain

$$\sqrt{\frac{8x^2y^4}{x^6y}} = \sqrt{\frac{8y^3}{x^4}}$$ Simplify: $\dfrac{8x^2y^4}{x^6y} = \dfrac{8y^3}{x^4}$

$$= \frac{\sqrt{8y^3}}{\sqrt{x^4}}$$ Quotient Rule for Square Roots

$$= \frac{\sqrt{4\cdot 2y^2y}}{\sqrt{x^4}}$$ Factor: $8 = 4\cdot 2$, $y^3 = y^2y$

$$= \frac{\sqrt{4}\sqrt{y^2}\sqrt{2y}}{\sqrt{x^4}} \qquad \text{Product Rule for Square Roots}$$

$$= \frac{2y\sqrt{2y}}{x^2} \qquad \text{Simplify: } \sqrt{4}=2,\ \sqrt{y^2}=y,\ \sqrt{x^4}=x^2$$

(d) We first simplify the expression under the radical sign of $\sqrt{\dfrac{40x^3y^2z}{2x^{-5}y^{-3}}}$ and then simplify the resulting expression. We obtain

$$\sqrt{\frac{40x^3y^2z}{2x^{-5}y^{-3}}} = \sqrt{\frac{20x^3y^2z}{x^{-5}y^{-3}}} \qquad \text{Divide by } 2$$

$$= \sqrt{20x^3x^5y^2y^3z} \qquad \text{Convert to positive exponents}$$

$$= \sqrt{20x^8y^5z} \qquad \text{Simplify: } x^3x^5=x^8,\ y^2y^3=y^5$$

$$= \sqrt{4\cdot 5x^8y^4yz} \qquad \text{Factor: } 20=4\cdot 5,\ y^5=y^4y$$

$$= \sqrt{4}\sqrt{x^8}\sqrt{y^4}\sqrt{5yz} \qquad \text{Product Rule for Square Roots}$$

$$= 2x^4y^2\sqrt{5yz} \qquad \text{Simplify: } \sqrt{4}=2,\ \sqrt{x^8}=x^4,\ \sqrt{y^4}=y^2$$

Exercise Set 3.2

Simplify the expression. Assume that all variables represent positive numbers.

1. $\sqrt{12x^6}$ **2.** $\sqrt{32x^2y^{10}}$ **3.** $\sqrt{81x^9}$ **4.** $\sqrt{25x^5y^9}$

5. $\sqrt{24z^7}$ **6.** $\sqrt{18y^3z^{12}}$ **7.** $\sqrt{8x^5y^{17}}$ **8.** $\sqrt{3x^8y^5z^7}$

9. $\sqrt{\dfrac{20}{x^{20}}}$ **10.** $\sqrt{\dfrac{x^6}{y^{10}z^2}}$ **11.** $\sqrt{\dfrac{32x^3}{9y^2}}$ **12.** $\sqrt{\dfrac{y^9z^{13}}{w^{12}}}$

13. $\sqrt{72x^5y^9}$ **14.** $\sqrt{\dfrac{49}{8x^5}}$ **15.** $\sqrt{\dfrac{27y^8}{8z^3}}$ **16.** $\sqrt{\dfrac{52y^8}{45z^{12}}}$

17. $\sqrt{\dfrac{18x^5}{2x^{-2}}}$ **18.** $\sqrt{\dfrac{x^5yz^8}{x^2y^{-5}}}$ **19.** $\sqrt{\dfrac{9x^{-2}y^6}{45x^{-7}y^2}}$ **20.** $\sqrt{\dfrac{33x^2y^6}{9x^{-3}}}$

Find the value of the algebraic expression at the specified values of its variable or variables in simplified form without a calculator. Simplify the expression before evaluating. Check your answer using a calculator.

21. $\sqrt{x^3y^4}$; $x=3, y=2$

22. $\sqrt{x^3y^5}$; $x=4, y=7$

23. $\sqrt{44x^5}$; $x=7$

24. $\sqrt{98x^6}$; $x=2$

25. $\sqrt{\dfrac{18x^5}{2x^{-2}}}$; $x=4$

26. $\sqrt{\dfrac{x^3y^{-4}}{x^7y^{-2}}}$; $x=3, y=5$

3.3. Roots and Rational Exponents

KYOTE Standards: *CA 1, CA 4*

Roots

Square roots are special cases of nth roots. We separate the definition of nth roots into two types, one for n an even positive integer such as 2 and the other for n an odd positive integer such as 3. These two types of nth roots have slightly different properties.

Definition 1. If n is an even positive integer, the nth root of a nonnegative real number a is that unique nonnegative real number, denoted by $\sqrt[n]{a}$, such that $\left(\sqrt[n]{a}\right)^n = a$. When $n = 2$, we write $\sqrt[2]{a}$ as $\sqrt{a}$, the square root of a.

It follows that $\sqrt[4]{81} = 3$ since $3^4 = 81$, and that $\sqrt[6]{64} = 2$ since $2^6 = 64$. On the other hand, notice that $\sqrt{-4}$ is not defined since there is no real number whose square is -4. This is true since the square of any real number is always positive or zero. Similarly, $\sqrt[4]{-81}$ is not defined since there is no real number whose fourth power is -81. *Thus when n is even, the nth root of a negative number is not defined for this same reason*.

Definition 2. If n is an odd positive integer greater than 1, the nth root of a real number a is that unique real number, denoted by $\sqrt[n]{a}$, such that $\left(\sqrt[n]{a}\right)^n = a$.

The nth root of an odd integer n, unlike an even integer n, is defined for *any* real number. We have, for example, $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$ and $\sqrt[5]{-32} = -2$ since $(-2)^5 = -32$.

Rational Exponents

To define what we mean by a *rational*, or *fractional*, exponent, we begin with rational exponents of the form $1/n$ where n is a positive integer.

Definition 3. For any rational number $1/n$, where n is a positive integer, we define

$$a^{1/n} = \sqrt[n]{a}$$

Here a is a real number and, if n is even, $a \geq 0$.

We can then extend this definition to all rational exponents.

Definition 4. For any rational number m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m \quad \textit{or equivalently} \quad a^{m/n} = \sqrt[n]{a^m}$$

Here a is a real number and, if n is even, $a \geq 0$.

The Properties of Exponents for integer exponents we considered in Section 3.1 also hold for rational exponents.

Properties of Exponents

If p and q are rational numbers in lowest terms with positive integer denominators, then

Property	**Examples**	
1. $a^p a^q = a^{p+q}$	$5^{3/2}5^{1/2} = 5^{3/2+1/2} = 5^2 = 25$	$8^{-1/3}8^{2/3} = 8^{-1/3+2/3} = 8^{1/3} = 2$
2. $\dfrac{a^p}{a^q} = a^{p-q}$	$\dfrac{4^{7/2}}{4^{3/2}} = 4^{7/2-3/2} = 4^2 = 16$	$\dfrac{7^{5/3}}{7} = 7^{5/3-1} = 7^{2/3} = \sqrt[3]{7^2}$
3. $\left(a^p\right)^q = a^{pq}$	$\left(2^{3/4}\right)^4 = 2^{3/4\cdot 4} = 2^3 = 8$	$(8)^{-2/3} = \left((8)^{1/3}\right)^{-2} = (-2)^{-2} = \dfrac{1}{4}$
4. $(ab)^p = a^p b^p$	$(16\cdot 3)^{1/2} = 16^{1/2}\cdot 3^{1/2} = 4\sqrt{3}$	$(8\cdot 4)^{1/3} = 8^{1/3}\cdot 4^{1/3} = 2\sqrt[3]{4}$
5. $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$	$\left(\dfrac{16}{9}\right)^{1/2} = \dfrac{16^{1/2}}{9^{1/2}} = \dfrac{4}{3}$	$\left(\dfrac{8}{27}\right)^{2/3} = \dfrac{8^{2/3}}{27^{2/3}} = \dfrac{(8^{1/3})^2}{(27^{1/3})^2} = \dfrac{4}{9}$

Each property holds for real numbers a and b for which the expressions on both sides of the equality are defined.

Note. The Product Rule for Square Roots can be written in exponential form $(ab)^{1/2} = a^{1/2}b^{1/2}$ as illustrated by the first example of Property 4. The Quotient Rule for Square Roots can be written in exponential form $\left(\dfrac{a}{b}\right)^{1/2} = \dfrac{a^{1/2}}{b^{1/2}}$ as illustrated by the first example of Property 5.

Rational exponents and their properties provide important ways to do calculations. The square root of a number can be calculated using the square root key on any calculator. However, if we want to compute the cube root or any other root of a number, or any rational power of a number, we must use rational exponents. For example, if we want to calculate the value of $\sqrt[3]{2}$, we must enter this number as $2^{1/3}$ since there is no cube root key on a calculator.

The following examples show how roots, rational exponents and the properties of exponents can be applied to evaluate numerical expressions and simplify algebraic expressions.

Example 1. Write each expression in radical form as an expression in exponential form and each expression in exponential form as an expression in radical form.

(a) $\sqrt[5]{x^2}$ **(b)** $\dfrac{1}{\sqrt{x^3}}$ **(c)** $3^{4/7}$ **(d)** $5^{-2/3}$

***Solution*. (a)** We write

$$\sqrt[5]{x^2} = \left(x^2\right)^{1/5} \quad \text{Convert to exponential form}$$

$$= x^{2/5} \quad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

Note: We could also use Definition 4 to write $\sqrt[5]{x^2} = x^{2/5}$ directly.

(b) We write

$$\frac{1}{\sqrt{x^3}} = \frac{1}{(x^3)^{1/2}} \quad \text{Convert to exponential form}$$

$$= \frac{1}{x^{3/2}} \quad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

(c) We write

$$3^{4/7} = (3^4)^{1/7} \quad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

$$= \sqrt[7]{3^4} \quad \text{Convert to radical form}$$

(d) We write

$$5^{-2/3} = \frac{1}{5^{2/3}} \quad \text{Property 2*: } \frac{a^p}{a^q} = a^{p-q}$$

$$= \frac{1}{(5^2)^{1/3}} \quad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

$$= \frac{1}{\sqrt[3]{5^2}} \quad \text{Convert to radical form}$$

*Note.** The expression $\dfrac{a^p}{a^q} = a^{p-q}$ in Property 2 gives $\dfrac{1}{a^q} = a^{-q}$ when $p = 0$. Hence $5^{-2/3} = \dfrac{1}{5^{2/3}}$.

Example 2. Find the value of the given expression.

(a) $\sqrt[3]{125}$ **(b)** $\sqrt[5]{\dfrac{1}{32}}$ **(c)** $9^{3/2}$ **(d)** $\left(-\dfrac{8}{27}\right)^{2/3}$

Solution. **(a)** We write $\sqrt[3]{125}$ in exponential form and use the properties of exponents to evaluate it.

$$\begin{aligned} \sqrt[3]{125} &= (125)^{1/3} && \text{Convert to exponential form} \\ &= \left(5^3\right)^{1/3} && \text{Factor: } 125 = 5^3 \\ &= 5 && \text{Property 3: } \left(a^p\right)^q = a^{pq} \end{aligned}$$

(b) We write $\sqrt[5]{\dfrac{1}{32}}$ in exponential form and use the properties of exponents to evaluate it.

$$\begin{aligned} \sqrt[5]{\frac{1}{32}} &= \left(\frac{1}{32}\right)^{1/5} && \text{Convert to exponential form} \\ &= \frac{(1)^{1/5}}{(32)^{1/5}} && \text{Property 5: } \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p} \\ &= \frac{(1)^{1/5}}{\left(2^5\right)^{1/5}} && \text{Factor: } 32 = 2^5 \\ &= \frac{1}{2} && \text{Property 3: } \left(a^p\right)^q = a^{pq} \end{aligned}$$

(c) We use the properties of exponents to write

$$\begin{aligned} 9^{3/2} &= \left(9^{1/2}\right)^3 && \text{Property 3: } \left(a^p\right)^q = a^{pq} \\ &= 3^3 && \text{Simplify: } 9^{1/2} = \sqrt{9} = 3 \\ &= 27 && \text{Simplify: } 3^3 = 27 \end{aligned}$$

Note. We could also use Property 3 to write $9^{3/2} = \left(9^3\right)^{1/2} = (729)^{1/2} = 27$ but most of us would need a calculator to find the square root of 729!

(d) We use the properties of exponents to write

$$\left(-\frac{8}{27}\right)^{2/3} = \frac{(-8)^{2/3}}{27^{2/3}} \qquad \text{Property 5: } \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$$

$$= \frac{\left((-8)^{1/3}\right)^2}{\left(27^{1/3}\right)^2} \qquad \text{Property 3: } \left(a^p\right)^q = a^{pq}$$

$$= \frac{(-2)^2}{3^2} \qquad \text{Simplify: } (-8)^{1/3} = 2,\ 27^{1/3} = 3$$

$$-\frac{4}{9} \qquad \text{Simplify: } (-2)^2 = 4,\ 3^2 = 9$$

Example 3. Simplify the given expression. Assume that all variables represent positive numbers.

(a) $\sqrt[3]{32x^6y^{17}}$ **(b)** $\sqrt[3]{\dfrac{81x^{13}}{y^{21}}}$ **(c)** $\sqrt[5]{\dfrac{x^{18}}{32}}$

***Solution*. (a)** We first write the expression in exponential form and use the properties of exponents to obtain

$$\sqrt[3]{32x^6y^{17}} = (32x^6y^{17})^{1/3} \qquad \text{Convert to exponential form}$$

$$= (32)^{1/3}(x^6)^{1/3}(y^{17})^{1/3} \qquad \text{Property 4: } (ab)^p = a^pb^p$$

$$= (8\cdot 4)^{1/3}x^2(y^{15}y^2)^{1/3} \qquad \text{Factor: } 32 = 8\cdot 4,\ y^{17} = y^{15}y^2, \text{ Property 3}$$

$$= 8^{1/3}4^{1/3}x^2(y^{15})^{1/3}(y^2)^{1/3} \qquad \text{Property 4: } (ab)^p = a^pb^p$$

$$= 2x^2y^5 4^{1/3}(y^2)^{1/3} \qquad 8^{1/3} = 2; \text{ Property 3: } \left(a^p\right)^q = a^{pq}$$

$$= 2x^2y^5(4y^2)^{1/3} \qquad \text{Property 4: } (ab)^p = a^pb^p$$

If we convert the answer to radical form, we obtain $\sqrt[3]{32x^6y^{17}} = 2x^2y^5\sqrt[3]{4y^2}$.

(b) We first write the expression in exponential form and use the properties of exponents to obtain

$$\sqrt[3]{\frac{81x^{13}}{y^{21}}} = \left(\frac{81x^{13}}{y^{21}}\right)^{1/3} \qquad \text{Convert to exponential form}$$

$$= \frac{(81x^{13})^{1/3}}{(y^{21})^{1/3}} \qquad \text{Property 5: } \left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$$

$$= \frac{(81)^{1/3}(x^{13})^{1/3}}{y^7}$$ Property 4: $(ab)^p = a^p b^p$, Property 3: $(a^p)^q = a^{pq}$

$$= \frac{(27\cdot 3)^{1/3}(x^{12}x)^{1/3}}{y^7}$$ Factor: $81 = 27\cdot 3$, $x^{13} = x^{12}x$

$$= \frac{3\cdot 3^{1/3}x^4 x^{1/3}}{y^7}$$ Property 4: $(ab)^p = a^p b^p$, Property 3: $(a^p)^q = a^{pq}$

$$= \frac{3x^4(3x)^{1/3}}{y^7}$$ Property 4: $(ab)^p = a^p b^p$

If we convert to radical form, we obtain $\sqrt[3]{\frac{81x^{13}}{y^{21}}} = \frac{3x^4\sqrt[3]{3x}}{y^7}$.

(c) We first write the expression in exponential form and use the properties of exponents to obtain

$$\sqrt[5]{\frac{x^{18}}{32}} = \left(\frac{x^{18}}{32}\right)^{1/5}$$ Convert to exponential form

$$= \frac{(x^{18})^{1/5}}{32^{1/5}}$$ Property 5: $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$

$$= \frac{(x^{15}x^3)^{1/5}}{2}$$ $(32)^{1/5} = 2$; Factor: $x^{18} = x^{15}x^3$

$$= \frac{x^3 x^{3/5}}{2}$$ Property 4: $(ab)^p = a^p b^p$, Property 3: $(a^p)^q = a^{pq}$

If we convert to radical form, we obtain $\sqrt[5]{\frac{x^{18}}{32}} = \frac{x^3\sqrt[5]{x^3}}{2}$.

Exercise Set 3.3

Simplify the expression. Assume that all variables represent positive numbers.

1. $\sqrt[3]{8x^{11}}$ **2.** $\sqrt[3]{16x^7}$ **3.** $\sqrt[3]{40x^6y^5}$ **4.** $\sqrt[3]{72x^{10}y^5}$

5. $\sqrt[3]{\frac{48}{y^{12}}}$ **6.** $\sqrt[3]{\frac{64x^{25}}{y^8}}$ **7.** $\sqrt[3]{\frac{54x^{18}}{y^{27}}}$ **8.** $\sqrt[3]{\frac{32x^{28}}{27y^{15}}}$

Write each radical expression as an exponential expression and each exponential expression as a radical expression.

9. $\frac{1}{\sqrt{3}}$ **10.** $\sqrt[3]{7^2}$ **11.** $\sqrt{5^3}$ **12.** $\frac{1}{\sqrt[3]{x^7}}$

13. $x^{2/3}$ **14.** $7^{-5/2}$ **15.** $x^{3/5}$ **16.** $x^{-5/3}$

Evaluate each expression without a calculator. Check your answer using a calculator.

17. $\sqrt[3]{\frac{8}{27}}$ **18.** $\sqrt[3]{-64}$ **19.** $\sqrt[5]{32}$ **20.** $\sqrt[4]{\frac{1}{16}}$

21. $\sqrt[3]{-\frac{54}{2}}$ **22.** $\frac{\sqrt[3]{-27}}{\sqrt[3]{8}}$ **23.** $\frac{\sqrt[5]{-3}}{\sqrt[5]{96}}$ **24.** $\frac{\sqrt[3]{5}}{\sqrt[3]{40}}$

25. $8^{1/3}$ **26.** $64^{1/4}$ **27.** $(-32)^{1/5}$ **28.** $81^{1/2}$

Find the value of the algebraic expression at the specified values of its variable or variables without using a calculator. Check your answer using a calculator.

29. $x^{2/3}; \quad x = 8$ **30.** $x^{3/4}; \quad x = 16$

31. $(x^3y)^{1/2}; \quad x = 2, y = 4$ **32.** $x^{4/3}; \quad x = 8$

33. $\left(\frac{x^4}{y^2}\right)^{3/2}; \quad x = 2, y = 3$ **34.** $(x^2 - y)^{3/2}; \quad x = 5, y = -7$

35. $(x^3 + y^3)^{5/2}; \quad x = 1, y = 2$ **36.** $(xy^2)^{-2/3}; \quad x = 2, y = -2$

37. $(x^2 + y^2)^{-1/2}; \quad x = 3, y = 4$ **38.** $\left(\frac{x^2}{y^2}\right)^{-5/2}; \quad x = 3, y = 2$

Chapter 4. Polynomials

4.1. Add and Subtract Polynomials

KYOTE Standards: *CR 8; CA 2*

Polynomials in one variable are algebraic expressions such as $3x^2 - 7x - 4$. In this example, the polynomial consists of three terms, $3x^2$, $-7x$ and -4. Each term is either a constant (a real number) such as -4 or constant multiplied by a power of the variable such as $3x^2$ and $-7x$.

Polynomials are classified by the number of terms they contain and by their degree. The *degree* of a polynomial is the highest power of the variable. The polynomial $3x^2 - 7x - 4$, for example, has degree 2. A polynomial such as $2x^3$ with only one term is called a *monomial*. A polynomial such as $x^5 - 8$ with two terms is called a *binomial*. A polynomial such as $x^4 - 5x^3 + 1$ is called a *trinomial*. The following table provides more examples about how polynomials are classified.

Polynomial	Type	Terms	Degree
$3x^2 - x + 5$	Trinomial	$3x^2$, $-x$, 5	2
$7 - x^4$	Binomial	7, $-x^4$	4
$x^5 - 2x^3 + 6x^2 - 1$	Four Terms	x^5, $-2x^3$, $6x^2$, -1	5
$9x^8$	Monomial	$9x^8$	8
-12	Monomial	-12	0

Polynomials can be added, subtracted, multiplied and factored. A key property of real numbers called the *Distributive Property* is the foundation for all these operations on polynomials.

Distributive Property
If a, b and c are real numbers, then

$$a(b+c) = ab + ac \quad \textit{and} \quad (b+c)a = ba + ca = ab + ac$$

The addition and subtraction of polynomials involves "combining like terms" and this technique, in turn, is an application of the Distributive Property. For example, when we add $2x^2$ and $3x^2$, we use the distributive property to obtain

$$2x^2 + 3x^2 = (2+3)x^2 = 5x^2$$

The multiplication of a polynomial by a constant or another monomial is also an application of the Distributive Property. For example, the expansions

$$6(2x^2-3x+8)=12x^2-18x+48 \qquad \textit{and} \qquad 2x^3(x^2-4)=2x^5-8x^3$$

are both applications of the Distributive Property. The following example further illustrates how the Distributive Property is applied to add and subtract polynomials.

Example 1. Perform the indicated operations and simplify.

(a) $3(x^2-x+5)-2(4x^2-5x-3)$

(b) Subtract $3x^2-5x+4$ from x^3-x^2+6x

(c) $3x^2(x-4)+4(x^3+7x^2-5)$

(d) $x^2-\frac{3}{8}x^2+\frac{5}{6}x^2$

Solution. **(a)** We first apply the Distributive Property to multiply each term in the first polynomial by 3 and each term in the second polynomial by -2 and then collect like terms to obtain

$3(x^2-x+5)-2(4x^2-5x-3)$	Given polynomial
$=3x^2-3x+15-8x^2+10x+6$	Distributive Property
$=3x^2-8x^2-3x+10x+15+6$	Group like terms
$=(3-8)x^2+(-3+10)x+(15+6)$	Collect like terms
$=-5x^2+7x+21$	Simplify

It is not necessary to include the next to last step. We included it to emphasize how the Distributive Property is used to collect like terms.

(b) We begin by writing the subtraction problem in algebraic form as

$$(x^3-x^2+6x)-(3x^2-5x+4)$$

We apply the Distributive Property to multiply each term of the second polynomial by -1 and collect like terms to obtain

$(x^3-x^2+6x)-(3x^2-5x+4)$	Given polynomial
$=x^3-x^2+6x-3x^2+5x-4$	Distributive Property
$=x^3-x^2-3x^2+6x+5x-4$	Group like terms
$=x^3-4x^2+11x-4$	Collect like terms

(c) We first use the Distributive Property to multiply each term in the first polynomial by $3x^2$ and each term in the second polynomial by 4 and collect like terms to obtain

$3x^2(x-4)+4(x^3+7x^2-5)$	Given polynomial

$= 3x^3 - 12x^2 + 4x^3 + 28x^2 - 20$ Distributive Property
$= 7x^3 + 16x^2 - 20$ Collect like terms

(d) All terms in this polynomial $x^2 - \frac{3}{8}x^2 + \frac{5}{6}x^2$ are like terms since they are all multiples of x^2. Two of the three coefficients are fractions and appear intimidating. However, if we realize that the coefficient of the first term is 1 and that we can use the Distributive Property, then we understand that this problem is an exercise in adding fractions. We obtain

$x^2 - \frac{3}{8}x^2 + \frac{5}{6}x^2$ Given polynomial

$= \left(1 - \frac{3}{8} + \frac{5}{6}\right)x^2$ Distributive Property

$= \left(\frac{24}{24} - \frac{9}{24} + \frac{20}{24}\right)x^2$ Write each fraction with LCD 24

$= \frac{35}{24}x^2$ Add fractions

Exercise Set 4.1

Perform the indicated operations and simplify.

1. $(4x-3)-(7x-5)$

2. $9-3(1-4y)$

3. $(2x^2-5x+8)+(5x^2-7x-12)$

4. $(3x^2-5x+8)-(8x^2-7x-11)$

5. $(x^2-3x)-(x^3-x^2-6x+1)$

6. $2(3t+1)-4(1-3t)$

7. $2x(x^2-1)+4(x^3-5x)$

8. $-3(5t^2-4)-t(2t-1)$

9. $4(x^2-x+5)-3(x^2-4x-3)$

10. $y^2(1-y)+3y(y^2-5y+1)$

11. Add x^3-3x^2+5x-7 and x^4-4x^3-x+9

12. Add $1-3y+5y^2$ and $4+8y-12y^2-6y^3$

13. Subtract x^2-2x-5 from $6x^2+5x-9$

14. Subtract $-5x^2+7x-3$ from x^3+4x^2-3

15. $\frac{1}{5}x+\frac{3}{5}x$

16. $\frac{7}{3}x^3-\frac{2}{3}x^3$

17. $t+\frac{1}{4}t$

18. $\frac{3}{8}t+2t$

19. $\frac{1}{3}x+\frac{3}{4}x+x$

20. $\frac{3}{8}x^2-\frac{1}{6}x^2$

21. $\frac{5x}{6}+\frac{3x}{4}$

22. $3x^2-\frac{x^2}{3}+\frac{4x^2}{5}$

23. $\left(x^2-\frac{3}{2}x+\frac{1}{4}\right)-\left(\frac{2}{5}x^2-\frac{5}{8}x+\frac{1}{3}\right)$

24. $\left(\frac{2x^2}{5}-\frac{x}{2}+\frac{5}{6}\right)-\left(\frac{1}{2}x^2-\frac{x}{8}-\frac{2}{3}\right)$

4.2. Multiply Polynomials

KYOTE Standards: *CR 9; CA 2*

We saw in Section 4.1 how the Distributive Property was applied to find the product of a monomial and a polynomial. For example, the product $2x^2(x^3-8x)$ is expanded using the Distributive Property to obtain

$$2x^2(x^3-8x)=2x^2\cdot x^3-2x^2\cdot 8x=2x^5-16x^3$$

The Distributive Property is used when multiplying any two polynomials. The most commonly encountered is the product of two binomials. For example, suppose we are asked to find $(2x-3)(4x+5)$. We think of $4x+5$ as a monomial and apply the Distributive Property and properties of exponents to obtain

$(2x-3)(4x+5)$	$=2x(4x+5)-3(4x+5)$	Distributive Property
	$=2x\cdot 4x+2x\cdot 5-3\cdot 4x-3\cdot 5$	Distributive Property
	F ***O*** ***I*** ***L***	
	$=8x^2+10x-12x-15$	Multiply factors
	$=8x^2-2x-15$	Collect like terms

This gives the well-known ***FOIL*** method. We find the product of the ***F***irst terms in each binomial, add the product of the ***O***uter terms, add the product of the ***I***nner terms, and finally add the product of the ***L***ast terms. We complete the expansion by finding the indicated products and combining like terms.

You are encouraged to become proficient with the ***FOIL*** method to the point where you can carry out the calculations mentally without writing down any intermediate steps. This ability helps us factor polynomials efficiently as we will see in Section 4.3. But it is also important to understand that the ***FOIL*** method is based on the Distributive Property and *only* applies when multiplying two binomials.

The Distributive Property, on the other hand, can be applied to find the product of any two polynomials. For example, suppose we want to find the product of a binomial and a trinomial in two variables such as $(a+b)(a^2-ab+b^2)$. We apply the Distributive Property and properties of exponents to obtain

$(a+b)(a^2-ab+b^2)$	$=a(a^2-ab+b^2)+b(a^2-ab+b^2)$	Distributive Property
	$=(a^3-a^2b+ab^2)+(a^2b-ab^2+b^3)$	Distributive Property
	$=a^3+b^3$	Combine like terms

Additional examples of polynomial multiplication are discussed below.

Example 1. Perform the indicated operations and simplify.

(a) $3x^2y(x^4 - 2x^2y^2 + 5y^4)$ **(b)** $(2x + 3y)^2$

(c) $5(2a^2 - 3b^2)(4a^2 + b^2)$ **(d)** $(x + 2y)(x^2 - 3xy + y^2)$

Solution. **(a)** We apply the Distributive Property and the properties of exponents to obtain

$$3x^2y(x^4 - 2x^2y^2 + 5y^4) = 3x^6y - 6x^4y^3 + 15x^2y^5 \qquad \text{Distributive Property}$$

(b) If we think about what $(2x + 3y)^2$ means, then this calculation is a simple application of the ***FOIL*** method. We have

$$\begin{aligned} (2x+3y)^2 &= (2x+3y)(2x+3y) && \text{Definition of } (2x+3y)^2 \\ &= 2x \cdot 2x + 2x \cdot 3y + 3y \cdot 2x + 3y \cdot 3y && \textbf{\textit{FOIL}} \\ &= 4x^2 + 6xy + 6xy + 9y^2 && \text{Multiply factors} \\ &= 4x^2 + 12xy + 9y^2 && \text{Collect like terms} \end{aligned}$$

Note: The incorrect squaring of a binomial is a very common error that you should try to avoid. Students often *incorrectly* write $(2x + 3y)^2 = (2x)^2 + (3y)^2 = 4x^2 + 9y^2$. The error occurs by confusing *terms* with *factors*. Students apply the property of exponents for *factors* $(2x \cdot 3y)^2 = (2x)^2(3y)^2 = 4x^2 \cdot 9y^2$ not realizing that they are dealing with *terms*.

(c) There are two strategies that could be used to expand the product $5(2a^2 - 3b^2)(4a^2 + b^2)$. We can first multiply the two binomials, and then multiply the result by 5, or we could multiply one of the binomials by 5, and then multiply the resulting binomials. We choose the first strategy to obtain

$$\begin{aligned} 5(2a^2 - 3b^2)(4a^2 + b^2) &= 5\left(8a^4 + 2a^2b^2 - 12a^2b^2 - 3b^4\right) && \textbf{\textit{FOIL}} \\ &= 5\left(8a^4 - 10a^2b^2 - 3b^4\right) && \text{Collect like terms} \\ &= 40a^4 - 50a^2b^2 - 15b^4 && \text{Distributive Property} \end{aligned}$$

(d) We cannot use the ***FOIL*** method because we have a binomial multiplied by a trinomial. Instead, we apply the Distributive Property, properties of exponents and combine like terms to obtain

$$\begin{aligned} (x+2y)(x^2 - 3xy + y^2) &= x(x^2 - 3xy + y^2) + 2y(x^2 - 3xy + y^2) && \text{Distributive Property} \\ &= (x^3 - 3x^2y + xy^2) + (2x^2y - 6xy^2 + 2y^3) && \text{Distributive Property} \end{aligned}$$

$$= x^3 - x^2y - 5xy^2 + 2y^3$$ Collect like terms

Exercise Set 4.2

Perform the indicated operations and simplify.

1. $3x^2(x^4 - 2x^3 - x + 5)$

2. $x^2y^3(x^4 - x^2y^2 + y^4)$

3. $2a^5(a^3 - 3a^2b - 4b^3)$

4. $(x+3)(x-2)$

5. $(3t-2)(5t+4)$

6. $(t-2)(5t+6)$

7. $2(3x+1)(x+5)$

8. $(x^2+y^2)(x^2-y^2)$

9. $(5y-4)(y-3)$

10. $3(2x-1)(4x+7)$

11. $(3x-2)^2$

12. $(3x-2)(3x-2)$

13. $(x^2-2y^3)(x^2-2y^3)$

14. $(x^2-2y^3)^2$

15. $(1-3x)^2$

16. $(a+b)^2$

17. $(y+2)(y^2-3y+4)$

18. $(a-b)(a^2+ab+b^2)$

19. $(x+2)^3$

20. $(x^3+2y^5)(x^3-2y^5)$

21. $(x^3+2y^5)(x^3+2y^5)$

22. $(x+3y)(x^2-6xy+9y^2)$

23. $(x+1)(x+1)^2$

24. $(2a-3)(a^2+a+1)$

25. $(2x-1)^3$

26. $(a+b)^3$

4.3. Factor Polynomials

KYOTE Standards: *CR 11; CA 5*

The Distributive Property

When we multiplied polynomials, we took expressions in parentheses in the form $a(b+c)$ and expanded them to remove the parentheses using the Distributive Property to obtain $ab+ac$. When we factor a polynomial in the form $ab+ac$, we factor out the common factor a from both terms and use the Distributive Property to obtain $a(b+c)$.

Factoring Out the GCF

We discussed the greatest common factor, the *GCF*, of two positive integers in Section 1.2 and how it could be found by factoring each integer into a product of its prime factors. We extended this concept to finding the *GCF* of two algebraic expressions and this extension is important in factoring out the GCF of two or more terms in a polynomial.

Suppose we are asked to factor the polynomial $12a^2b^3+18a^5b$ in two variables a and b. We begin by finding the *GCF* of $12a^2b^3$ and $18a^5b$. The *GCF* of 12 and 18 is 6 and the *GCF* of a^2b^3 and a^5b is a^2b. Thus the *GCF* of $12a^2b^3+18a^5b$ is $6a^2b$, the greatest common factor of both terms. We can then factor out $6a^2b$ from both terms and apply the Distributive Property to obtain the desired factorization.

$$\begin{aligned} 12a^2b^3+18a^5b &= 6a^2b\cdot 2b^2+6a^2b\cdot 3a^3 && \text{Factor } 6a^2b \text{ from each term} \\ &= 6a^2b(2b^2+3a^2) && \text{Factor out } 6a^2b \end{aligned}$$

Example 1. Find the greatest common factor (*GCF*) of the terms in the expression. Write the expression by factoring out the *GCF* in each of its terms and then use the distributive law to write the expression in factored form.

(a) $9x^3y-12x^2y^2$ **(b)** $8x^3-12x^2+36x$ **(c)** $15a^2(b-1)-18a^3(b-1)$

Solution. **(a)** The *GCF* of the terms $9x^3y$ and $12x^2y^2$ is $3x^2y$. We factor out $3x^2y$ from both terms and apply the Distributive Property to obtain

$$\begin{aligned} 9x^3y-12x^2y^2 &= 3x^2y\cdot 3x-3x^2y\cdot 4y && \text{Factor } 3x^2y \text{ from each term} \\ &= 3x^2y(3x-4y) && \text{Factor out } 3x^2y \end{aligned}$$

(b) The *GCF* of the three terms $8x^3$, $12x^2$ and $36x$ is $4x$. We factor out $4x$ from both terms and apply the Distributive Property to obtain

$$8x^3 - 12x^2 + 36x = 4x \cdot 2x^2 - 4x \cdot 3x + 4x \cdot 9 \quad \text{Factor } 4x \text{ from each term}$$
$$= 4x(2x^2 - 3x + 9) \quad \text{Factor out } 4x$$

(c) The *GCF* of $42a^2(b-1)$ and $18a^3(b-1)$ is $6a^2(b-1)$. We factor out $6a^2(b-1)$ from both terms and apply the Distributive Property to obtain

$$42a^2(b-1) - 18a^3(b-1) = 6a^2(b-1) \cdot 7 - 6a^2(b-1) \cdot 3a$$
$$\text{Factor } 6a^2(b-1) \text{ from each term}$$
$$= 6a^2(b-1)(7-3a) \quad \text{Factor out } 6a^2(b-1)$$

Factoring Trinomials with Leading Coefficient 1

We use a trial and error method to factor trinomials of the form $x^2 + bx + c$ where b and c are integers.

Example 2. Factor: **(a)** $x^2 - 6x + 8$ **(b)** $x^2 + 9x + 8$ **(c)** $x^2 + 4x + 8$

Solution. All three trinomials have a constant term of 8 and so the same trial and error process can be used to factor them. The following table shows the possible factors of 8, the corresponding possible binomial factors and the middle term resulting when these factors are multiplied.

Factors of 8	**Possible Binomial Factors**	**Middle Term**
$8, 1$	$(x+8)(x+1)$	$9x$
$-8, -1$	$(x-8)(x-1)$	$-9x$
$4, 2$	$(x+4)(x+2)$	$6x$
$-4, -2$	$(x-4)(x-2)$	$-6x$

(a) We see from the table that $x^2 - 6x + 8 = (x-4)(x-2)$ since $-6x$ is the middle term we seek.

(b) We see from the table that $x^2 + 9x + 8 = (x+8)(x+1)$ since $9x$ is the middle term we seek.

(c) The table lists the *possible* binomial factors of $x^2 + 4x + 8$. Since none of these factors has a middle term $4x$, $x^2 + 4x + 8$ cannot be factored.

Example 3. Factor: **(a)** $x^2 + 4x - 12$ **(b)** $x^2 - x - 12$ **(c)** $x^2 + 2x - 12$

Solution. All three trinomials have a constant term of -12 and so the same trial and error process can be used to factor them. The only difference between this example and Example 1 is that there are more possible binomial factors. The following table shows the possible factors of -12, the corresponding possible binomial factors and the middle term resulting when these factors are multiplied.

Factors of -12	Possible Binomial Factors	Middle Term
$12, -1$	$(x+12)(x-1)$	$11x$
$-12, 1$	$(x-12)(x+1)$	$-11x$
$6, -2$	$(x+6)(x-2)$	$4x$
$-6, 2$	$(x-6)(x+2)$	$-4x$
$4, -3$	$(x+4)(x-3)$	x
$-4, 3$	$(x-4)(x+3)$	$-x$

(a) We see from the table that $x^2+4x-12=(x+6)(x-2)$ since $4x$ is the middle term we seek.

(b) We see from the table that $x^2-x-12=(x-4)(x+3)$ since $-x$ is the middle term we seek.

(c) The table lists the *possible* binomial factors of $x^2+2x-12$. Since none of these factors has a middle term $2x$, $x^2+2x-12$ cannot be factored.

Note. The trial and error process of factoring trinomials requires more work than the reverse process of multiplying two binomials. The examples above give a systematic way to implement the trail and error process. However, you should not in general have to write out an entire table to factor trinomials of this kind. Instead, much of the work in finding the possible binomial factors and the middle terms can be done mentally with practice.

Factoring Trinomials with Leading Coefficient Not 1

We use a trial and error method discussed in the next two examples.

Example 4. Factor: **(a)** $3x^2-13x-10$ **(b)** $3x^2+7x-10$ **(c)** $3x^2-5x-10$

Solution. All three trinomials have the same first term and last term and so the same trial and error process can be used to factor them. The following table shows the factors of 3, the coefficient of x^2, and -10, the constant term.

Factors of 3	Factors of -10	Possible Binomial Factors	Middle Term
$3, 1$	$10, -1$	$(3x+10)(x-1)$ $(3x-1)(x+10)$	$7x$ $29x$
$3, 1$	$-10, 1$	$(3x-10)(x+1)$ $(3x+1)(x-10)$	$-7x$ $-29x$
$3, 1$	$5, -2$	$(3x+5)(x-2)$ $(3x-2)(x+5)$	$-x$ $13x$
$3, 1$	$-5, 2$	$(3x-5)(x+2)$ $(3x+2)(x-5)$	x $-13x$

Notice that for each pair of factors of -10 there are two possible binomial factors. For example, the factors $5,-2$ correspond to the two binomial factors $(3x+5)(x-2)$ and $(3x-2)(x+5)$. The large number of possible factors makes it all the more important that you can calculate the middle term quickly. We suggest that you go through eight binomial factors in the table, calculate the middle term mentally, and check with the answer in the table.

(a) We see from the table that $3x^2-13x-10=(3x+2)(x-5)$ since $-13x$ is the middle term we seek.

(b) We see from the table that $3x^2+7x-10=(3x+10)(x-1)$ since $7x$ is the middle term we seek.

(c) The table lists the *possible* binomial factors of $3x^2-5x-10$. Since none of these factors has a middle term $-5x$, $3x^2-5x-10$ cannot be factored.

Example 5. Factor: **(a)** $6x^2-13x+5$ **(b)** $6x^2-11x+5$ **(c)** $6x^2+9x+5$

Solution. All three trinomials have the same first term and last term and so the same trial and error process can be used to factor them. The following table shows the factors of 6, the coefficient of x^2, and 5, the constant term.

Factors of 6	**Factors of** 5	**Possible Binomial Factors**	**Middle Term**
$6,1$	$5,1$	$(6x+5)(x+1)$ $(6x+1)(x+5)$	$11x$ $31x$
$6,1$	$-5,-1$	$(6x-5)(x-1)$ $(6x-1)(x-5)$	$-11x$ $-31x$
$3,2$	$5,1$	$(3x+5)(2x+1)$ $(3x+1)(2x+5)$	$13x$ $17x$
$3,2$	$-5,-1$	$(3x-5)(2x-1)$ $(3x-1)(2x-5)$	$-13x$ $-17x$

(a) We see from the table that $6x^2-13x+5=(3x-5)(2x-1)$ since $-13x$ is the middle term we seek.

(b) We see from the table that $6x^2-11x+5=(6x-5)(x-1)$ since $-11x$ is the middle term we seek.

(c) The table lists the *possible* binomial factors of $6x^2+9x+5$. Since none of these factors has a middle term $9x$, $6x^2+9x+5$ cannot be factored.

Factoring Difference of Squares

A formula can be used to factor a few special polynomials. The most important such formula is the difference of squares.

Difference of Squares Formula

$$A^2 - B^2 = (A-B)(A+B)$$

Example 6. Factor: **(a)** $25x^2 - 9$ **(b)** $x^2 - 4y^2$ **(c)** $a^4 - 16b^4$

Solution. **(a)** We recognize $25x^2 - 9 = (5x)^2 - (3)^2$ as the difference of squares with $A = 5x$ and $B = 3$. We use the difference of squares formula to obtain

$25x^2 - 9 = (5x)^2 - (3)^2$	Write each term as a square
$= (5x-3)(5x+3)$	Difference of Squares Formula

(b) We recognize $x^2 - 4y^2 = (x)^2 - (2y)^2$ as the difference of squares with $A = x$ and $B = 2y$. We use the difference of squares formula to obtain

$x^2 - 4y^2 = (x)^2 - (2y)^2$	Write each term as a square
$= (x-2y)(x+2y)$	Difference of Squares Formula

(c) We recognize $a^4 - 16b^4 = (a^2)^2 - (4b^2)^2$ as the difference of squares with $A = a^2$ and $B = 4b$. We use the difference of squares formula to obtain

$a^4 - 16b^4 = (a^2)^2 - (4b^2)^2$	Write each term as a square
$= (a^2 - 4b^2)(a^2 + 4b^2)$	Difference of Squares Formula

We then observe that $a^2 - 4b^2 = (a-2b)(a+2b)$ is also the difference of squares and we write the complete factorization as

$$a^4 - 4b^4 = (a-2b)(a+2b)(a^2+4b^2)$$

Factoring a Polynomial Completely

A complete factorization of a polynomial often requires more than one step. When we factor a polynomial, *we first factor out the greatest common factor*, then inspect the result to see if further factoring is possible.

Example 7. Factor the given polynomial completely.

(a) $8x^3y^2 - 2xy^4$ **(b)** $3x^2 - 6xy - 24y^2$

(c) $4a^3 + 10a^2 - 6a$ **(d)** $x^2(y^2-1) - 25(y^2-1)$

Solution. **(a)** We first factor out the *GCF* $2xy^2$ of the two terms in $8x^3y^2 - 2xy^4$ and then recognize that the resulting factor $4x^2 - y^2$ is the difference of squares to obtain

$8x^3y^2 - 2xy^4 = 2xy^2(4x^2 - y^2)$ Factor out $2xy^2$
$= 2xy^2(2x - y)(2x + y)$ Factor: $4x^2 - y^2$

(b) We first factor out the *GCF* 3 of the three terms in $3x^2 - 6xy - 24y^2$ and then recognize that the resulting factor $x^2 - 2xy - 8y^2$ is a trinomial that we can factor. We obtain

$3x^2 - 6xy - 24y^2 = 3(x^2 - 2xy - 8y^2)$ Factor out 3
$= 3(x - 4y)(x + 2y)$ Factor: $x^2 - 2xy - 8y^2$

(c) We first factor out the *GCF* $2a$ of the three terms in $4a^3 + 10a^2 - 6a$ and then recognize that the resulting factor $2a^2 + 5a - 3$ is a trinomial that we can factor. We obtain

$4a^3 + 10a^2 - 6a = 2a(2a^2 + 5a - 3)$ Factor out $2a$
$= 2a(2a - 1)(a + 3)$ Factor: $2a^2 + 5a - 3$

(d) We first realize that the *binomial* $y^2 - 1$ is a common factor (and indeed the greatest common factor) of the two terms in $x^2(y^2 - 1) - 25(y^2 - 1)$. We factor it out to obtain

$x^2(y^2 - 1) - 25(y^2 - 1) = (x^2 - 25)(y^2 - 1)$ Factor out $y^2 - 1$

We then recognize that both $x^2 - 25$ and $y^2 - 1$ are the difference of squares. We complete the factorization to obtain

$x^2(y^2 - 1) - 25(y^2 - 1) = (x^2 - 25)(y^2 - 1)$ Factor out $y^2 - 1$
$= (x - 5)(x + 5)(y - 1)(y + 1)$ Factor $x^2 - 25$, $y^2 - 1$

Exercise Set 4.3

Find the greatest common factor (GCF) of the terms in the expression. Write the expression by factoring out the GCF in each of its terms and then use the distributive law to write the expression in factored form.

1. $4x^3 + 8x$

2. $10y + 15$

3. $24x^3 - 8x^5$

4. $a^2b^5 + a^3b^3$

5. $3a^2b - 7a^3b^4$

6. $27x^5y^4 - 18x^2y^3$

7. $45t^5 - 30t^3$

8. $5s^2t^2 - st$

9. $6y^2(x+1) + 8y(x+1)$

10. $x(5x-1) + 3(5x-1)$

11. $4a(a-1) - 7(a-1)$

12. $18x^2y^2(x-y) + 12xy^3(x-y)$

13. $9a^2b + 6ab^3 + 15a^4b^2$

14. $x^4 + 3x^3 - 18x^2$

15. $8x^2 - 20x + 32$

16. $12a^6b^2 - 9a^5b^3 + 15a^4b^4$

17. $24x^2(x+2) + 30x(x+2) + 15(x+2)$

18. $9x^3(2y-1) - 8x^5(2y-1) + 18x^7(2y-1)$

Factor the trinomial.

19. $x^2 + 2x - 3$

20. $x^2 - 5x + 4$

21. $x^2 - x - 6$

22. $x^2 - 7x + 10$

23. $x^2 - 5x - 24$

24. $x^2 - 11x + 18$

25. $x^2 - 8xy + 7y^2$

26. $x^2 - 4y^2$

27. $a^2 + ab - 12b^2$

28. $x^2 - 14xy + 24$

29. $9s^2 - 16t^2$

30. $a^2 - 4ab - 21b^2$

31. $t^2 + 3t - 40$

32. $x^2 - 8xy - 48y^2$

33. $y^2 + 2y - 24$

34. $x^2 - 81$

Complete the factorization.

35. $2x^2 + 5x - 3 = (2x \qquad)(x \qquad)$

36. $6x^2 + 7x - 5 = (3x \qquad)(2x \qquad)$

37. $6x^2 - 11x - 2 = (6x \qquad)(x \qquad)$

38. $8x^2 - 2x - 15 = (4x \qquad)(2x \qquad)$

39. $8x^2 - 26x + 15 = (4x \qquad)(2x \qquad)$

40. $5x^2 - 13x + 6 = (5x \qquad)(x \qquad)$

41. $12x^2 + 16x - 3 = (6x \qquad)(2x \qquad)$

42. $12x^2 + 41x + 35 = (4x \qquad)(3x \qquad)$

43. $5x^2 - 23x + 24 = (5x \qquad)(x \qquad)$

44. $5x^2 + 14x - 24 = (5x \qquad)(x \qquad)$

Factor the trinomial.

45. $2x^2 + 5x - 3$

46. $2y^2 - 19y - 10$

47. $3a^2 + 8a + 5$

48. $5x^2 + 7x - 6$

49. $3t^2 + 13t - 10$

50. $3y^2 - 2y - 1$

51. $4x^2 - 4x - 3$

52. $4x^2 - 11x - 3$

53. $9a^2 - 18a - 16$

54. $4y^2 + 12y + 9$

55. $6x^2 - x - 12$

56. $2y^2 - 11y + 14$

57. $6b^2 + 7b - 3$

58. $4s^2 - 9s + 2$

59. $15x^2 + 16x + 4$

60. $8t^2 + 5t - 22$

61. $2x^2 + xy - 6y^2$

62. $2s^2 - 11st + 5t^2$

63. $6a^2 - 7ab + 2b^2$

64. $9x^2 + 24xy + 16y^2$

Factor the expression completely.

65. $12x^3 + 18x$

66. $6a^4b^2 - 9a^3b^3$

67. $x^2 - 8x + 15$

68. $9x^2 - 36x - 45$

69. $3x^2 - 27$

70. $2t^2 + 5t + 3$

71. $10x^4 - 35x^3 + 15x^2$

72. $2x^4 - 18x^2$

73. $a^4b^3 - a^2b^5$

74. $12y^3 + 50y^2 + 28y$

75. $6x^3 + 45x^2 + 21x$

76. $12a^2 + 36ab + 27b^2$

77. $2x^3y^2 + 13x^2y^2 + 15xy^2$

78. $14t^5 - 38t^4 + 20t^3$

79. $x^6y^2 - 9x^4y^4$

80. $6x^2 - xy - 12y^2$

81. $6x^2 - xy - 12y^2$

82. $3x^2 + 10xy - 24y^2$

83. $5ab^2x^2 - 10ab^2x - 15ab^2$

84. $24a^2 - 18ab + 3b^2$

85. $x^4 + 5x^2 + 6$

86. $2x^4 - 5x^2 + 3$

87. $x^4 - y^4$

88. $a^6(a+1)^2 + a^7(a+1)$

89. $(x-2)(x+5)^2 + (x-2)^2(x+5)$

90. $x^2(x^2-1) - 9(x^2-1)$

91. $9x^4 - 49$

92. $5a^3 - 125a$

93. $x^2(x-3) - 4(x-3)$

94. $a^2(x-y) - b^2(x-y)$

95. $5x^4 - 80y^4$

96. $4x^2 + 24xy + 36y^2$

Chapter 5. Rational Expressions

5.1. Simplify Rational Expressions

KYOTE Standards: *CR 13; CA 7*

Definition 1. A **rational expression** is the quotient $\frac{P}{Q}$ of two polynomials P and Q in one or more variables, where $Q \neq 0$.

Some examples of rational expressions are:

$$\frac{x}{x^2+2x-4} \qquad \frac{ab^2}{c^4} \qquad \frac{x^3-y^3}{x^2+y^2}$$

Rational expressions in algebra are closely related to *rational* numbers in arithmetic as their names and definitions suggest. Recall that a rational number is the quotient $\frac{p}{q}$ of two integers p and q, where $q \neq 0$, as stated in Section 1.3. The procedure used to simplify a rational expression by dividing out the greatest common factor of the numerator and denominator is the same as the procedure used to reduce a rational number (a fraction) to lowest terms. The procedures used to multiply, divide, add and subtract rational expressions are the same as the corresponding procedures used to multiply, divide, add and subtract rational numbers.

Example 1. Find the greatest common factor (*GCF*) of the numerator and the denominator of each rational expression. Write the expression by factoring out the *GCF* in both the numerator and the denominator. Then divide out the *GCF* to write the rational expression in simplified form.

(a) $\frac{18a^2b^5c^3}{12a^4b}$ **(b)** $\frac{9x^2(2x-1)^3}{24x^5(2x-1)}$

Solution. **(a)** The *GCF* of the numerator and denominator of $\frac{18a^2b^5c^3}{12a^4b}$ is $6a^2b$. We factor out the *GCF* of the numerator and denominator and divide it out to obtain

$$\frac{18a^2b^5c^3}{12a^4b} = \frac{6a^2b \cdot 3b^4c^3}{6a^2b \cdot 2a^2}$$ Factor $6a^2b$ from numerator and denominator

$$= \frac{3b^4c^3}{2a^2}$$ Divide out (cancel) $6a^2b$

(b) The *GCF* of the numerator and denominator of $\frac{9x^2(2x-1)^3}{24x^5(2x-1)}$ is $3x^2(2x-1)$. We factor out the *GCF* of the numerator and denominator and divide it out to obtain

$$\frac{9x^2(2x-1)^3}{24x^5(2x-1)} = \frac{3x^2(2x-1)\cdot 3(2x-1)^2}{3x^2(2x-1)\cdot 8x^3}$$ Factor $3x^2(2x-1)$ from numerator and denominator

$$= \frac{3(2x-1)^2}{8x^3}$$ Divide out (cancel) $3x^2(2x-1)$

Example 2. Simplify the given rational expression.

(a) $\frac{3a^2b}{3a^2b+6ab}$ **(b)** $\frac{x^2-x-6}{3x^2+5x-2}$ **(c)** $\frac{a^2b^3-a^2b}{a^2b+a^2}$ **(d)** $\frac{y^2-x^2}{x^2-xy}$

Solution. **(a)** We factor the denominator of $\frac{3a^2b}{3a^2b+6ab}$ and divide out the *GCF* $3ab$ of numerator and denominator to obtain

$$\frac{3a^2b}{3a^2b+6ab} = \frac{3a^2b}{3ab(a+2)}$$ Factor $3ab$ from numerator and denominator

$$= \frac{a}{a+2}$$ Divide out (cancel) $3ab$

Note: *Remember that you can only divide out, or cancel, an expression if it is a* ***factor*** *of the numerator and denominator of a rational expression.* A common mistake in this case is to assume that $3a^2b$ is a *factor* of the denominator instead of a term and "cancel" it out to obtain the incorrect simplification $\frac{1}{6ab}$.

(b) We employ what we have learned about factoring polynomials to factor the numerator and denominator polynomials of $\frac{x^2-x-6}{3x^2+5x-2}$ and divide out the common factor $x+2$ to obtain

$$\frac{x^2-x-6}{3x^2+5x-2} = \frac{(x-3)(x+2)}{(3x-1)(x+2)}$$ Factor numerator and denominator polynomials

$$= \frac{x-3}{3x-1}$$ Divide out (cancel) $x+2$

(c) We factor the numerator and denominator polynomials of $\frac{a^2b^3 - a^2b}{a^2b + a^2}$ in two steps and divide out the common factor after each step to obtain

$$\frac{a^2b^3 - a^2b}{a^2b + a^2} = \frac{a^2(b^3 - b)}{a^2(b+1)}$$ Factor a^2 from numerator and denominator polynomials

$$= \frac{b(b^2 - 1)}{b+1}$$ Divide out (cancel) a^2; Factor $b^3 - b$

$$= \frac{b(b-1)(b+1)}{b+1}$$ Factor $b^2 - 1 = (b-1)(b+1)$

$$= b(b-1)$$ Divide out (cancel) $b+1$

(d) We factor the numerator and denominator polynomials of $\frac{y^2 - x^2}{x^2 - xy}$ in two steps and divide out the common factor after the second step to obtain

$$\frac{y^2 - x^2}{x^2 - xy} = \frac{(y-x)(y+x)}{x(x-y)}$$ Factor numerator and denominator polynomials

$$= \frac{-(x-y)(y+x)}{x(x-y)}$$ Write $y - x = -(x-y)$

$$= -\frac{y+x}{x}$$ Divide out (cancel) $x - y$

Exercise Set 5.1

Find the greatest common factor (GCF) of the numerator and the denominator of each rational expression. Write the expression by factoring out the GCF in both the numerator and the denominator. Then divide out the GCF to write the rational expression in simplified form.

1. $\frac{45a^3b^4}{9a^5b}$

2. $\frac{8a^{10}b^3}{6a^5b}$

3. $\frac{3x(x-1)^2}{5x^2(x-1)}$

4. $\frac{24x^3y^5}{30y^3z^2}$

5. $\frac{x^2(x+3)^3}{x^7(x+3)(2x-1)}$

6. $\frac{15(x+1)^3(x-1)}{48(x+1)^5(2x+3)}$

Simplify the rational expression.

7. $\dfrac{18x^4y^7}{24x^8y^4z}$

8. $\dfrac{6(x+4)^3(x-2)^2}{30(x+4)^2}$

9. $\dfrac{4(a-b)(a+b)^2}{7(b-a)(a+b)^2}$

10. $\dfrac{(x-2)(3x+5)^2}{(2-x)(3x+5)^3}$

11. $\dfrac{x^3+3x^2}{x^2+2x^4}$

12. $\dfrac{3x^2-15x}{12x-60}$

13. $\dfrac{a^2b^2+a^2b^4}{a^2b^2+a^4b^2}$

14. $\dfrac{x^2y}{x^2y+x^4y^2}$

15. $\dfrac{x^3yz+xy^3z+xyz^3}{x^2y^2z^2}$

16. $\dfrac{6t^4-18t^3}{4t^2-12t}$

17. $\dfrac{t^3-2t^2+t}{t^2-t}$

18. $\dfrac{2a^2b^2-10a^6b^8}{2a^2b^2}$

19. $\dfrac{x^2-4}{x+2}$

20. $\dfrac{x^2+4x+3}{x+1}$

21. $\dfrac{x^2+6x+8}{x^2+5x+4}$

22. $\dfrac{(a-3)^2}{a^2-9}$

23. $\dfrac{x^2+2x-3}{x^2+x-6}$

24. $\dfrac{6x+12}{x^2+5x+6}$

25. $\dfrac{4x^2-4}{12x^2+12x-24}$

26. $\dfrac{4y^3+4y-8y}{2y^3+4y-6y}$

27. $\dfrac{y^2-y-12}{y^2+5y+6}$

28. $\dfrac{2x^2+5x-3}{3x^2+11x+6}$

29. $\dfrac{x^2+2xy+y^2}{3x^2+2xy-y^2}$

30. $\dfrac{2x^2+6xy+4y^2}{4x^2-4y^2}$

31. $\dfrac{6x+12}{4x^2+6x-4}$

32. $\dfrac{x^7+4x^6+3x^5}{x^4+3x^3+2x^2}$

33. $\dfrac{3x^2+12x+12}{9x^2-36}$

34. $\dfrac{t^3+t^2}{t^2-1}$

35. $\dfrac{t^2-3t-18}{2t^2+5t+3}$

36. $\dfrac{2t^4-t^3-6t^2}{2t^2-7t+6}$

5.2. Multiply and Divide Rational Expressions

KYOTE Standards: *CR 12; CA 6*

The procedures used to multiply and divide rational expressions are the same as those used to multiply and divide fractions. For example, if A, B, C and D are polynomials with $B \neq 0$ and $C \neq 0$, then

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{A \cdot C}{B \cdot D} \qquad \textit{and} \qquad \frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{A \cdot D}{B \cdot C}$$

Once these operations are performed, the only remaining task is to simply the resulting rational expression as we did in Section 5.1. A few examples should illustrate this process.

Example 1. Perform the multiplication or division of the given rational expressions and simplify.

(a) $\dfrac{x^2+2x-8}{x^2-9} \cdot \dfrac{x^2+3x}{x^2-5x+6}$ **(b)** $\dfrac{x^5}{5x-10} \div \dfrac{x^2}{2x-4}$

(c) $\dfrac{2a^2-ab-b^2}{a^2+6ab+9b^2} \cdot \dfrac{a^2+4ab+3b^2}{b-a}$

Solution. **(a)** We factor the numerators and denominators of both rational expressions in the product $\dfrac{x^2+2x-8}{x^2-9} \cdot \dfrac{x^2+3x}{x^2-5x+6}$ and simplify the resulting rational expression to obtain

$$\frac{x^2+2x-8}{x^2-9} \cdot \frac{x^2+3x}{x^2-5x+6}$$ Given product of rational expressions

$$= \frac{(x-2)(x+4)}{(x-3)(x+3)} \cdot \frac{x(x+3)}{(x-2)(x-3)}$$ Factor polynomials

$$= \frac{x(x+4)}{(x-3)^2}$$ Divide out (cancel) $x-2$, $x+3$

(b) We invert the second expression $\dfrac{x^2}{2x-4}$ in the quotient $\dfrac{x^5}{5x-10} \div \dfrac{x^2}{2x-4}$, multiply it by the first expression $\dfrac{x^5}{5x-10}$ and simplify to obtain

$$\frac{x^5}{5x-10} \div \frac{x^2}{2x-4}$$ Given quotient of rational expressions

$$= \frac{x^5}{5x-10} \cdot \frac{2x-4}{x^2}$$ Invert $\frac{x^2}{2x-4}$ and multiply

$$= \frac{x^5}{5(x-2)} \cdot \frac{2(x-2)}{x^2}$$ Factor $2x-4$, $5x-10$

$$= \frac{2x^3}{5}$$ Divide out (cancel) $x-2$, x^2

(c) We factor the numerators and denominators of both rational expressions in the product $\frac{2a^2-ab-b^2}{a^2+6ab+9b^2} \cdot \frac{a^2+4ab+3b^2}{b-a}$ in two steps and divide out the common factor after each step to obtain

$$\frac{2a^2-ab-b^2}{a^2+6ab+9b^2} \cdot \frac{a^2+4ab+3b^2}{b-a}$$ Given product of rational expressions

$$= \frac{(2a+b)(a-b)}{(a+3b)(a+3b)} \cdot \frac{(a+3b)(a+b)}{b-a}$$ Factor polynomials

$$= \frac{(2a+b)(a-b)}{(a+3b)} \cdot \frac{a+b}{b-a}$$ Divide out (cancel) $a+3b$

$$= \frac{-(2a+b)(b-a)}{a+3b} \cdot \frac{a+b}{b-a}$$ Write $a-b=-(b-a)$

$$= -\frac{(2a+b)(a+b)}{a+3b}$$ Divide out (cancel) $b-a$

Exercise Set 5.2

Perform the multiplication or division and simplify.

1. $\frac{x^2}{4y} \cdot \frac{2x^3}{y^2}$

2. $-\frac{y}{8} \cdot \frac{10}{y^3}$

3. $\frac{a^2}{a+2} \cdot \frac{6a+12}{a^2}$

4. $\frac{8a-6}{5a+20} \cdot \frac{2a+8}{4a-3}$

5. $\frac{x^3}{y} \div \frac{x^5}{y^2}$

6. $\frac{2a+3}{a^3} \div \frac{6a+9}{a^4}$

7. $\frac{x-1}{(x+3)^2} \div \frac{1-x}{(x+3)^3}$

8. $\frac{4x-1}{3x+2} \cdot \frac{9x^2+6x}{1-4x}$

9. $\frac{3x^2}{x^2-9}\cdot\frac{x+3}{12x}$

10. $\frac{2x^2+7x-4}{2x^2-3x+1}\cdot\frac{3-x}{x-3}$

11. $\frac{x^2-x-6}{x^2-1}\cdot\frac{x+1}{x-3}$

12. $\frac{x^2+5x+6}{x^2+2x}\cdot\frac{x^3+x}{x^2+4x+3}$

13. $\frac{x^2y+3xy^2}{x^2-9y^2}\cdot\frac{x^2-2xy-3y^2}{5x^2y}$

14. $\frac{2x^2+3x+1}{x^2+2x-15}\div\frac{x^2+6x+5}{2x^2-7x+3}$

15. $\frac{x^4}{x+2}\div\frac{x^3}{x^2+4x+4}$

16. $\frac{3x^2+2x-1}{x^2-1}\cdot\frac{x^2-2x+1}{3x^2-7x+2}$

17. $\frac{2a^2-ab-b^2}{a^2-2ab+b^2}\cdot\frac{2a^2+ab-3b^2}{2a^2+3ab+b^2}$

18. $\frac{x^2-2x-15}{x^2-4x-5}\cdot\frac{x^2+8x+7}{x^2+7x+12}$

5.3. Add and Subtract Rational Expressions

KYOTE Standards: *CR 12; CA 6*

The procedures used to add and subtract rational expressions are the same as those used to add and subtract fractions. Two rational expressions with the same denominator are relatively easy to add and subtract. For example, if A, B and C are polynomials with $C \neq 0$, then

$$\frac{A}{C}+\frac{B}{C}=\frac{A+B}{C} \qquad \textit{and} \qquad \frac{A}{C}-\frac{B}{C}=\frac{A-B}{C}$$

The procedure is more difficult to carry out if the denominators are different. For example, suppose A, B, C and D are polynomials with $C \neq 0$ and $D \neq 0$. To add the rational expressions $\frac{A}{C}$ and $\frac{B}{D}$, we must first find a common denominator, CD in this case. We then find equivalent expressions for $\frac{A}{C}$ and $\frac{B}{D}$ with this denominator and add to obtain

$$\frac{A}{C}+\frac{B}{D}=\frac{AD}{CD}+\frac{BC}{DC}=\frac{AD+BC}{CD}$$

In practice, it is important to find the *least common denominator*, or *LCD*, because otherwise the algebra becomes messy and it is difficult to reduce the rational expression that is obtained. In Section 1.2, we observed that the *LCD* of two or more fractions is the *least common multiple*, or *LCM*, of their denominators. We use this same approach to find the *LCD* of two or more rational expressions.

Example 1. Suppose the polynomials given are denominators of rational expressions. Find their least common denominator (*LCD*).

(a) $6x^3y^4, 8x^5y$ **(b)** $x^2-9, x^2-2x-15$ **(c)** $5(a+1)^2, 16(a+1)^3, 10(a+1)$

Solution. **(a)** To find the *LCD* of the denominators $6x^3y^4$ and $8x^5y$, we factor $6=2\cdot 3$ and $8=2^3$ into products of powers of prime numbers. We then examine these denominators in factored form:

$$2\cdot 3x^3y^4, \quad 2^3x^5y$$

We view the variables x and y as prime numbers and we take the largest power of each "prime" factor 2, 3, x and y in the two expressions to form the *LCD* as we did in Section 1.2. The *LCD* of $6x^3y^4$ and $8x^5y$ is therefore

$$2^3\cdot 3x^5y^4=24x^5y^4$$

(b) To find the LCD of the denominators x^2-9 and $x^2-2x-15$, we factor them to obtain $x^2-9=(x-3)(x+3)$ and $x^2-2x-15=(x-5)(x+3)$. The *LCD* of x^2-9 and $x^2-2x-15$ is therefore the product

$$(x-3)(x+3)(x-5)$$

Note. The product $(x-3)(x+3)^2(x-5)$ is *a common denominator* but not *the least common denominator*.

(c) To find the *LCD* of the denominators $5(a+1)^2$, $16(a+1)^3$ and $10(a+1)$, we factor $16=2^4$ and $10=2\cdot 5$ into products of powers of primes and examine these denominators in factored form:

$$5(a+1)^2, \quad 2^4(a+1)^3, \quad 2\cdot 5(a+1)$$

We take the largest power of each "prime" factor 2, 5 and $a+1$, and multiply them together to obtain their *LCD*

$$2^4\cdot 5(a+1)^3=80(a+1)^3$$

Example 2. Find the *LCD* of the given pair of rational expressions. Express each rational expression in the pair as an equivalent rational expression with the *LCD* as its denominator.

(a) $\dfrac{3}{4a^2b}, \dfrac{5}{6ab^3}$ **(b)** $\dfrac{x}{x^2-x-2}, \dfrac{x+4}{x^2+3x-10}$

Solution. **(a)** The *LCD* of the denominators $4a^2b$ and $6ab^3$ is $12a^2b^3$. Thus $12a^2b^3$ is a multiple of both $4a^2b$ and $6ab^3$, and we can write

$$12a^2b^3=4a^2b\cdot 3b^2 \qquad\qquad 12a^2b^3=6ab^3\cdot 2a$$

We can then write $\dfrac{3}{4a^2b}$ and $\dfrac{5}{6ab^3}$ as equivalent rational expressions with the same denominator $12a^2b^3$.

$$\frac{3}{4a^2b}=\frac{3\cdot 3b^2}{4a^2b\cdot 3b^2}=\frac{9b^2}{12a^2b^3} \qquad\qquad \frac{5}{6ab^3}=\frac{5\cdot 2a}{6ab^3\cdot 2a}=\frac{10a}{12a^2b^3}$$

(b) We must factor the denominator polynomials $x^2-x-2=(x-2)(x+1)$ and $x^2+3x-10=(x-2)(x+5)$ to find their *LCD*. Their *LCD* is therefore $(x-2)(x+1)(x+5)$.

We can then write $\frac{x}{x^2-x-2}$ and $\frac{x+4}{x^2+3x-10}$ as equivalent rational expressions with the same denominator $(x-2)(x+1)(x+5)$.

$$\frac{x}{x^2-x-2}=\frac{x}{(x-2)(x+1)}=\frac{x(x+5)}{(x-2)(x+1)(x+5)}$$

$$\frac{x+4}{x^2+3x-10}=\frac{x+4}{(x-2)(x+5)}=\frac{(x+4)(x+1)}{(x-2)(x+5)(x+1)}$$

Example 3. Perform the addition or subtraction and simplify. Identify the *LCD* in each case.

(a) $\frac{5x}{6}+\frac{3x}{8}$ **(b)** $\frac{1}{2a}+\frac{1}{3a^2}$ **(c)** $\frac{2}{x-2}-\frac{3}{x+1}$

Solution. **(a)** The *LCD* of the denominators 6 and 8 is 24. Thus 24 is a multiple of both 6 and 8, and we can write $24=6\cdot 4$ and $24=8\cdot 3$. We write both $\frac{5x}{6}$ and $\frac{3x}{8}$ as equivalent expressions with the same denominator of 24 and add to obtain

$$\frac{5x}{6}+\frac{3x}{8}=\frac{5x\cdot 4}{6\cdot 4}+\frac{3x\cdot 3}{8\cdot 3}$$ Write each term as an equivalent expression with *LCD* 24

$$=\frac{20x}{24}+\frac{9x}{24}$$ Simplify

$$=\frac{29x}{24}$$ Add

(b) The *LCD* of the denominators $2a$ and $3a^2$ is $6a^2$. Thus $6a^2$ is a multiple of both $2a$ and $3a^2$, and we can write $6a^2=2a\cdot 3a$ and $6a^2=3a^2\cdot 2$. We write both $\frac{1}{2a}$ and $\frac{1}{3a^2}$ as equivalent expressions with the same denominator of $6a^2$ and add to obtain

$$\frac{1}{2a}+\frac{1}{3a^2}=\frac{1\cdot 3a}{2a\cdot 3a}+\frac{1\cdot 2}{3a^2\cdot 2}$$ Write each term as an equivalent expression with *LCD* $6a^2$

$$=\frac{3a}{6a^2}+\frac{2}{6a^2}$$ Simplify

$$=\frac{3a+2}{6a^2}$$ Add

(c) The *LCD* of the denominators $x-2$ and $x+1$ is $(x-2)(x+1)$. We write both $\frac{2}{x-2}$ and $\frac{3}{x+1}$ as equivalent expressions with the same denominator of $(x-2)(x+1)$, subtract and simplify to obtain

$$\frac{2}{x-2}-\frac{3}{x+1}=\frac{2(x+1)}{(x-2)(x+1)}-\frac{3(x-2)}{(x+1)(x-2)}$$ Write each term as an equivalent expression with *LCD* $(x-2)(x+1)$

$$=\frac{2(x+1)-3(x-2)}{(x-2)(x+1)}$$ Subtract

$$=\frac{2x+2-3x+6}{(x-2)(x+1)}$$ Expand numerator expressions

$$=\frac{-x+8}{(x-2)(x+1)}$$ Collect like terms

Example 4. Perform the addition or subtraction and simplify. Identify the *LCD* in each case.

(a) $2y+\frac{3}{y+1}$ **(b)** $\frac{1}{x^2+4x+4}-\frac{x+1}{x^2-4}$ **(c)** $\frac{3}{t}-\frac{2}{t+2}+\frac{4}{t^2+2t}$

Solution. **(a)** The term $2y$ is a rational expression with denominator 1. The *LCD* of the denominators 1 and $y+1$ is $y+1$. We write both $2y$ and $\frac{3}{y+1}$ as equivalent expressions with the same denominator of $y+1$ and add to obtain

$$2y+\frac{3}{y+1}=\frac{2y(y+1)}{y+1}+\frac{3}{y+1}$$ Write each term as an equivalent expression with *LCD* $y+1$

$$=\frac{2y^2+2y+3}{y+1}$$ Expand $2y(y+1)$ and add

(b) We must factor the denominator polynomials $x^2+4x+4=(x+2)^2$ and $x^2-4=(x-2)(x+2)$ to find their *LCD* $(x+2)^2(x-2)$. We write both $\frac{1}{x^2+4x+4}$ and $\frac{x+1}{x^2-4}$ as equivalent expressions with the same denominator of $(x+2)^2(x-2)$, subtract and simplify to obtain

$$\frac{1}{x^2+4x+4}-\frac{x+1}{x^2-4}=\frac{1}{(x+2)^2}-\frac{x+1}{(x-2)(x+2)}$$ Factor denominator polynomials

$$=\frac{x-2}{(x+2)^2(x-2)}-\frac{(x+1)(x+2)}{(x-2)(x+2)(x+2)}$$ Write each term as an equivalent expression with *LCD* $(x+2)^2(x-2)$

$$=\frac{x-2-(x+1)(x+2)}{(x-2)(x+2)^2}$$ Subtract

$$=\frac{x-2-(x^2+3x+2)}{(x-2)(x+2)^2}$$ Multiply $(x+1)(x+2)=x^2+3x+2$

$$=\frac{-x^2-2x-4}{(x-2)(x+2)^2}$$ Collect like terms

(c) We factor $t^2+2t=t(t+2)$ and we see that the *LCD* of the denominators t, $t+2$ and $t(t+2)$ is $t(t+2)$. We write $\frac{3}{t}$, $\frac{2}{t+2}$ and $\frac{4}{t(t+2)}$ as equivalent expressions with the same denominator of $t(t+2)$, subtract and add the numerators, and simplify to obtain

$$\frac{3}{t}-\frac{2}{t+2}+\frac{4}{t^2+2t}=\frac{3}{t}-\frac{2}{t+2}+\frac{4}{t(t+2)}$$ Factor $t^2+2t=t(t+2)$

$$=\frac{3(t+2)}{t(t+2)}-\frac{2t}{(t+2)t}+\frac{4}{t(t+2)}$$ Write each term as an equivalent expression with *LCD* $t(t+2)$

$$=\frac{3t+6-2t+4}{t(t+2)}$$ Multiply $3(t+2)=3t+6$; subtract and add

$$=\frac{t+10}{t(t+2)}$$ Collect like terms

Exercise Set 5.3

Suppose the expressions given are denominators of rational expressions. Find their least common denominator (LCD).

1. $x^4y^5,\ x^2y^7z$

2. $2a^3b^5, 3a^6b^2$

3. $12(x-1), 9(x-1)^3$

4. $(x+2)^3(x+3), (x+3)^2(x+4), (x+4)^5$

5. $x^2+5x+6,\ (x+2)^2$

6. $15x^2(y+1),\ 21x(y+1)^3$

7. $x^2-25,\ x^2+8x+15$

8. $t(t^2-1), t^3(t+1),\ t^2(t-1)$

Write each pair of rational expressions as equivalent rational expressions with their LCD as the denominator for both.

9. $\dfrac{7}{2x^2},\ \dfrac{5}{3x^2}$

10. $\dfrac{1}{4ab^2}, \dfrac{1}{a^3b}$

11. $\dfrac{1}{12x^2y^3},\ \dfrac{1}{18x^5y}$

12. $\dfrac{x+1}{4xy}, \dfrac{5y}{6x^2}$

13. $\dfrac{3}{x(x-1)}, \dfrac{7}{(x-1)^2}$

14. $\dfrac{x}{4x+4}, \dfrac{1}{x^2-1}$

15. $\dfrac{x}{x^2+4x+3}, \dfrac{x+5}{x^2+3x+2}$

16. $\dfrac{x+3}{x^2-x-2}, \dfrac{6x}{x^2-4x+4}$

Perform the addition or subtraction and simplify. Identify the LCD in each case.

17. $\dfrac{y}{7x^2}-\dfrac{4}{7x^2}$

18. $\dfrac{4}{(x+1)^2}+\dfrac{9}{(x+1)^2}$

19. $\dfrac{x}{4}+\dfrac{2x}{3}$

20. $\dfrac{1}{2a}+\dfrac{4}{3a}$

21. $\dfrac{1}{s}+\dfrac{1}{t}$

22. $\dfrac{2}{3x}-\dfrac{1}{6y}$

23. $\dfrac{3}{c^3}-\dfrac{4}{c^4}$

24. $\dfrac{5}{6a^2b}+\dfrac{2}{9ab^2}$

25. $\dfrac{3}{x}+\dfrac{5}{x+2}$

26. $\dfrac{1}{x-1}-\dfrac{1}{x+1}$

27. $5+\dfrac{4}{x-2}$

28. $\dfrac{3x}{x^2-4}-\dfrac{1}{x-2}$

29. $\dfrac{1}{2r}+\dfrac{1}{3s}+\dfrac{1}{4t}$

30. $\dfrac{2x-1}{3}-\dfrac{x-4}{5}$

31. $x-\frac{x}{x+7}$

32. $\frac{3}{x}+\frac{2}{x-1}-\frac{5}{x^2-x}$

33. $\frac{1}{a}+\frac{1}{a^2}+\frac{1}{a^3}$

34. $\frac{x}{x^3-x^2}+\frac{1}{x^3}$

35. $\frac{3}{x-5}+\frac{4}{5-x}$

36. $\frac{x}{x^2+x-6}+\frac{x+1}{x^2+7x+12}$

37. $\frac{2}{x^2+2x-15}-\frac{1}{x^2-5x+6}$

38. $\frac{7x-1}{x^2-9x+20}-\frac{1}{x-5}$

39. $\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{3}{x^2-1}$

40. $\frac{t+1}{2t^2+5t-3}+\frac{t+3}{2t^2-3t+1}$

Chapter 6. Linear Equations and Inequalities

6.1. Solve Linear Equations in one Variable

KYOTE Standards: *CR 14; CA 8*

An equation is a statement that two mathematical expressions are equal. The statement $7-2=5$ is an equation. The equations we study in algebra generally contain variables that represent numbers. For example,

$$2x-3=5$$

is an equation in the variable x. A value of x that makes the equation a true statement is called a *solution* of the equation. In this case, 4 is a solution to this equation because $2\cdot 4-3=5$ is a true statement. It is also the *only*, or *unique*, solution to this equation.

A *linear* equation in one variable such as our example $2x-3=5$ is an equation in which the largest power of the variable is 1. By contrast, the equation

$$x^2-2x-8=0$$

is a *quadratic* equation, not a *linear* equation, in one variable since the largest power of the variable is 2. Notice that in this case, there are *two* solutions, -2 and 4, to the quadratic equation since $(-2)^2-2(-2)-8=0$ and $(4)^2-2(4)-8=0$.

Two equations are *equivalent* if they have exactly the same solutions. For example, the following three equations are equivalent

$$2x-3=5, \quad 2x=8, \quad x=4$$

Moreover, these equations illustrate how we solve a linear equation by reducing it to simpler, equivalent equations until we have isolated the variable on one side and a number, the solution, on the other. There are two properties of equality that we use in this process.

Properties of Equality

1. Adding (or subtracting) the same number to both sides of an equation gives an equivalent equation.
2. Multiplying (or dividing) both sides of an equation by the same nonzero number gives an equivalent equation.

The following examples show how we solve a variety of linear equations.

Example 1. Solve the equation for x.

(a) $7x-20=2x-5$ **(b)** $2(3x-1)-5(2-x)=7$

***Solution*. (a)** We start with the assumption that x is a solution to the equation so that $7x-20=2x-5$. We then apply the properties of equality to get all terms involving x on one side of the equation and all terms that are numbers alone on the other side. Finally, we divide by the coefficient of x in the resulting equation to obtain the solution. The steps are as follows:

$7x-20=2x-5$	Given equation
$(7x-20)+20=(2x-5)+20$	Add 20
$7x=2x+15$	Simplify
$7x-2x=(2x+15)-2x$	Subtract $2x$
$5x=15$	Simplify
$\frac{5x}{5}=\frac{15}{5}$	Divide by 5
$x=3$	Simplify

We have shown by this series of steps is that *if* $7x-20=2x-5$, *then* $x=3$. This means that *if* there is a solution to the equation, *then* that solution must be 3. It is always a good idea to check that 3 is actually a solution by substituting 3 into to equation to show that $7(3)-20=2(3)-5$ is a true statement. Thus 3 is the *unique* solution of the equation; that is, it is a solution and it is the *only* solution of the equation.

(b) We first expand the expressions in parentheses to remove all parentheses. We then follow steps similar to those in part (a) to obtain the solution.

$2(3x-1)-5(2-x)=7$	Given equation
$6x-2-10+5x=7$	Expand
$11x-12=7$	Simplify
$(11x-12)+12=7+12$	Add 12
$11x=19$	Simplify
$\frac{11x}{11}=\frac{19}{11}$	Divide by 11
$x=\frac{19}{11}$	Simplify

In this case, it takes a little more effort to show that $\frac{19}{11}$ is a solution!

Example 2. Solve the equation for x.

(a) $\frac{2x-1}{3}=\frac{x+1}{4}$ **(b)** $x=0.15x+2000$

Solution. **(a)** The suggested strategy for solving equations involving fractions is to clear the fractions first. We do this by multiplying both sides of the equation by the *LCD* of the fractions involved. The process in this case is better known as "cross-multiplying." We obtain

$\frac{2x-1}{3}=\frac{x+1}{4}$	Given equation
$12\left(\frac{2x-1}{3}\right)=12\left(\frac{x+1}{4}\right)$	Multiply by 12
$4(2x-1)=3(x+1)$	Simplify
$8x-4=3x+3$	Expand
$5x=7$	Add 4; subtract $3x$
$x=\frac{7}{5}$	Divide by 5

(b) Equations like $x=0.15x+2000$ involving decimals occur frequently in applied problems. When we subtract $0.15x$ from both sides to obtain $x-0.15x=2000$, students often do not understand that the coefficient of x is 1 and we can use the distributive property to write

$$x-0.15x=(1-0.15)x=0.85x$$

We then solve the equation

$$0.85x=2000$$

We divide by 0.85 to obtain

$$x=2352.94$$

Example 3. Solve the equation $\frac{4}{5}x=\frac{2}{3}x+\frac{1}{6}$ for x.

Solution. We present two approaches to solving this equation. The first method of clearing fractions first is recommended, but you should also understand that the second method of placing all terms involving x on one side of the equation and all numbers on the other can be used even though working with the fractions may be difficult.

Method 1. We begin by clearing fractions. We multiply both sides of the equation by the *LCD* $30=2\cdot3\cdot5$ of the fractions involved. We obtain

$\frac{4}{5}x=\frac{2}{3}x+\frac{1}{6}$	Given equation

$$30\left(\frac{4}{5}x\right)=30\left(\frac{2}{3}x+\frac{1}{6}\right) \qquad \text{Multiply by } 30$$
$$24x=20x+5 \qquad \text{Expand and simplify}$$
$$4x=5 \qquad \text{Subtract } 20x$$
$$x=\frac{5}{4} \qquad \text{Divide by } 4$$

Method 2. We proceed as we would if the fractions involved were all integers. We obtain

$$\frac{4}{5}x=\frac{2}{3}x+\frac{1}{6} \qquad \text{Given equation}$$
$$\frac{4}{5}x-\frac{2}{3}x=\frac{1}{6} \qquad \text{Subtract } \frac{2}{3}x$$
$$\left(\frac{4}{5}-\frac{2}{3}\right)x=\frac{1}{6} \qquad \text{Collect like terms: distributive property}$$
$$\frac{2}{15}x=\frac{1}{6} \qquad \text{Subtract fractions}$$
$$\frac{15}{2}\cdot\left(\frac{2}{15}x\right)=\frac{15}{2}\cdot\frac{1}{6} \qquad \text{Multiply by } \frac{15}{2}$$
$$x=\frac{5}{4} \qquad \text{Simplify}$$

We note that the same answer is obtained using either method.

Example 4. The formula $F=\frac{9}{5}C+32$ expresses a temperature in degrees Celsius (C) as a temperature in degrees Fahrenheit (F). Find the temperature in degrees Celsius if the temperature in degrees Fahrenheit is 59.

Solution. We substitute 59 for F in the formula $F=\frac{9}{5}C+32$ to obtain $59=\frac{9}{5}C+32$. We then solve this equation for C to obtain

$$59=\frac{9}{5}C+32 \qquad \text{Given equation}$$
$$59-32=\frac{9}{5}C \qquad \text{Subtract } 32$$
$$27=\frac{9}{5}C \qquad \text{Simplify}$$

$$\frac{5}{9}(27) = \frac{5}{9}\left(\frac{9}{5}C\right) \qquad \text{Multiply by } \frac{5}{9}$$

$$15 = C \qquad \text{Simplify}$$

We conclude that a temperature of 59 degrees Fahrenheit corresponds to a temperature of 15 degrees Celsius.

Exercise Set 6.1

Solve the equation for x.

1. $3x - 5 = 13$

2. $6 - 2x = -8$

3. $5 = 3x - 9$

4. $-2 = 4 - x$

5. $3x + 7 = 5x - 11$

6. $9x + 4 = -4x - 7$

7. $2(3x - 7) = 8x + 10$

8. $4(x - 1) = 6(3 - 2x)$

9. $5 = 3 - (4 - 2x)$

10. $7x - (1 - 4x) = 12$

11. $5(2 - x) - 4(1 - 3x) = 6$

12. $5 - 2(3x - 8) = 3 - x$

13. $\frac{x}{5} = \frac{2}{7}$

14. $\frac{3}{4}x = 8$

15. $0.05x = 3$

16. $\frac{4x}{0.01} = \frac{200}{3}$

17. $\frac{2x}{7} = \frac{10}{3}$

18. $6\pi x = 27$

19. $4x - 3\pi = 7x + 6\pi$

20. $\frac{2x - 3}{5} = \frac{x + 1}{6}$

21. $\frac{3x - 5}{4} = x + \frac{1}{2}$

22. $\frac{5x - 1}{3} = 2x - 3$

23. $\frac{x + 1}{3} = \frac{7x + 2}{4}$

24. $4.35x + 32 = 5.75x - 17$

25. $0.05x = x - 190$

26. $0.07x + 0.05(1000 - x) = 65$

27. $25 = \frac{5}{3}(x - 12)$

28. $16 = \frac{4}{7}x - 8$

29. $\frac{1}{4}x - \frac{2}{3} = 1$

30. $\frac{7}{5}x - 7 = 2.8$

31. $\frac{2}{5}(x + 3) = x - 1$

32. $18\pi + 6\pi x = 420$

33. $x + \frac{1}{2}x = 5 - \frac{1}{3}x$

34. $\frac{2x}{3} + \frac{3x}{4} = 1$

35. The perimeter P of a rectangle of length l and width w is $P = 2l + 2w$. If the perimeter of a rectangle is 116 inches and its length is 31 inches, what is its width in inches?

36. The area A of a triangle of base b and height h is $A = \frac{1}{2}bh$. If the base of a triangle is 15 centimeters and its area is 165 square centimeters, what is its height in centimeters?

37. The volume V of a cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$. If the base radius is 5 inches and the volume is 700 cubic inches, what is the height in inches?

38. The formula for calculating a z-score for a sample score of x is $z = \frac{x - \overline{x}}{s}$ where $\overline{x}$ is the mean of the sample and s is the standard deviation. Find the sample score x if $\overline{x}$ =70, s =10 and $z = -2.4$.

39. The formula $C = \frac{5}{9}(F - 32)$ expresses a temperature in degrees Fahrenheit as a temperature in degrees Celsius. Find the temperature in degrees Fahrenheit if the temperature in degrees Celsius is **a)** 15 **b)** -10.

40. The surface area S of a cylinder is $S = 2\pi r^2 + 2\pi rh$ where r is the base radius and h is the height. If the base radius is 5 inches and the surface area is 250 square inches, what the height in inches?

6.2. Solve Multivariable Linear Equations for one of their Variables

KYOTE Standards: *CR 15; CA 9*

Many formulas and equations in mathematics, statistics, science, technology and business involving several variables require us to solve for one of the variables in terms of the others. One such example is the formula $C = \frac{5}{9}(F - 32)$ relating the temperature in degrees Fahrenheit F the temperature in degrees Celsius C. The objective is to solve for F in terms of C.

We use the same strategy to solve linear equations involving several variables that we use to solve linear equations in one variable. The following examples show how this is done.

Example 1. Solve the equation for the indicated variable.

(a) $3x + 2y = 7z;\ for\ x$ **(b)** $V = \frac{4}{3}\pi u^2 w;\ for\ w$ **(c)** $\frac{r}{a} - \frac{3t}{b} = 2;\ for\ t$

Solution. **(a)** We get all terms involving x ($3x$ in this case) on one side of $3x + 2y = 7z$ and all the remaining terms ($2y$ and $7z$) on the other side using the properties of equality. We obtain

$3x + 2y = 7z$	Given equation
$3x = 7z - 2y$	Subtract $2y$
$x = \frac{7z - 2y}{3}$	Divide by 3

(b) We first clear fractions by multiplying both sides of $V = \frac{4}{3}\pi u^2 w$ by 3 and then divide by the resulting coefficient $4\pi u^2$ of w to solve for w. We obtain

$V = \frac{4}{3}\pi u^2 w$	Given equation
$3V = 4\pi u^2 w$	Multiply by 3
$\frac{3V}{4\pi u^2} = w$	Divide by $4\pi u^2$

It is customary to write the variable we have solved for on the left side of the equation so that in this case $w = \frac{3V}{4\pi u^2}$.

(c) We first clear fractions by multiplying both sides of $\frac{r}{a}-\frac{3t}{b}=2$ by the *LCD* ab of the fractions involved. We then solve the resulting equation for t to obtain

$\frac{r}{a}-\frac{3t}{b}=2$	Given equation
$ab\left(\frac{r}{a}-\frac{3t}{b}\right)=2ab$	Multiply by ab
$br-3at=2ab$	Simplify
$br-2ab=3at$	Add $3at$; subtract $2ab$
$\frac{br-2ab}{3a}=t$	Divide by $3a$

Therefore $t=\frac{br-2ab}{3a}$.

Example 2. Solve the equation for the indicated variable.

(a) $ax-3(b-cx)=dx;\ for\ x$ **(b)** $\frac{x-a}{y+b}=\frac{cx}{d};\ for\ x$

Solution. **(a)** There are no fractions to clear so we begin by expanding $-3(b-cx)=-3b+3cx$ to remove the parentheses. We then get all terms involving x on one side of the equation and all other terms on the other side using the equality properties. We obtain

$ax-3(b-cx)=dx$	Given equation
$ax-3b+3cx=dx$	Expand
$ax+3cx-dx=3b$	Add $3b$; subtract dx
$(a+3c-d)x=3b$	Factor out x
$x=\frac{3b}{a+3c-d}$	Divide by $a+3c-d$

Note 1. It is important to understand *why* we get all terms involving x (ax, $3cx$ and dx) on one side of the equation and all other terms ($-3b$) on the other side. We do so because we can then factor out x using the distributive property and divide by the coefficient ($a+3c-d$) of x in order to solve for x.
Note 2. Solving a linear equation in several variables for one of its variables is usually considered more difficult for students than solving a linear equation in one variable. One reason is that "collecting like terms" (for example, $4x+3x-2x=(4+3-2)x=5x$) appears much easier than factoring $ax+3cx-dx=(a+3c-d)x$ even though both depend on the distributive property.

(b) We first clear fractions by multiplying both sides of $\frac{x-a}{y+b}=\frac{cx}{d}$ by the *LCD* $d(y+b)$ of the fractions involved. We then solve the resulting equation for x. We obtain

$\frac{x-a}{y+b}=\frac{cx}{d}$	Given equation
$d(y+b)\left(\frac{x-a}{y+b}\right)=d(y+b)\left(\frac{cx}{d}\right)$	Multiply by $d(y+b)$
$d(x-a)=(y+b)(cx)$	Simplify
$dx-da=ycx+bcx$	Expand
$dx-ycx-bcx=da$	Add da; subtract $ycx+bcx$
$(d-yc-bc)x=da$	Factor out x
$x=\frac{da}{d-yc-bc}$	Divide by $d-yc-bc$

Example 3. The formula $C=\frac{5}{9}(F-32)$ expresses a temperature in degrees Fahrenheit F as a temperature in degrees Celsius C. Solve this equation for F.

Solution. We first clear fractions by multiplying both sides of $C=\frac{5}{9}(F-32)$ by $\frac{9}{5}$. This approach is more efficient than multiplying by 9 although we get the same result using either approach. We then solve the resulting equation for F. We obtain

$C=\frac{5}{9}(F-32)$	Given equation
$\frac{9}{5}C=F-32$	Multiply by $\frac{9}{5}$; note $\frac{9}{5}\cdot\frac{5}{9}=1$
$\frac{9}{5}C+32=F$	Add 32

The formula $F=\frac{9}{5}C+32$ expresses a temperature in degrees Celsius C as a temperature in degrees Fahrenheit F.

Exercise Set 6.2

Solve the equation for the indicated variable.

1. $2x+3y=6$; *for y*

2. $5x=8x-7z$; *for x*

3. $ax+by=c$; *for y*

4. $V=\frac{4}{5}xyz$; *for z*

5. $\frac{3}{2}ab=9$; *for a*

6. $PV=nRT$; *for R*

7. $L=\frac{3}{5}F+9$; *for F*

8. $s=\frac{a-rz}{1-r}$; *for z*

9. $a^2x=bx+c$; *for x*

10. $bx-(c-x)=a+d$; *for x*

11. $a-(3b-cy)=dy$; *for y*

12. $\frac{3a}{c}=\frac{a-1}{d}$; *for a*

13. $\frac{x-a}{5}=\frac{y-b}{3}$; *for x*

14. $\frac{2x-3}{c}=\frac{x+1}{d}$; *for x*

15. $R=\frac{4}{9}(T-12)$; *for T*

16. $3x=2y(1-x)$; *for x*

17. $\frac{3}{4}x+\frac{2}{3}y=5$; *for x*

18. $a[b-(1-3cx)]=dx$; *for x*

19. $\frac{x+3}{a}+\frac{y+1}{b}=1$; *for x*

20. $z=\frac{ax-b}{c}$; *for x*

21. The perimeter P of a rectangle of length l and width w is $P=2l+2w$. Solve this equation for w.

22. The area A of a triangle of base b and height h is $A=\frac{1}{2}bh$. Solve this equation for b.

23. The circumference C of a circle of radius r is $C=2\pi r$. Solve this equation for r.

24. The volume V of a cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Solve this equation for h.

25. The formula $F = \frac{9}{5}C + 32$ expresses a temperature in degrees Celsius C as a temperature in degrees Fahrenheit F. Solve this equation for C.

26. The formula for calculating a z-score for a sample score of x is $z = \frac{x - \bar{x}}{s}$ where $\bar{x}$ is the mean of the sample and s is the standard deviation. Solve this equation for x.

27. The length L of a spring with natural length L_0 and with a weight of W pounds hanging from it is $L = \frac{4}{7}W + L_0$. Solve this equation for W.

28. The formula $F = \frac{GmM}{r^2}$ is Newton's Law of Gravitation, where F is the force of attraction between two bodies, one of mass m and the other of mass M, a distance r apart and where G is the gravitational constant. Solve this equation for m.

29. The volume V of a pyramid with square base of side length L and height h is $V = \frac{L^2 h}{3}$. Solve this equation for h.

30. The surface area S of a cylinder is $S = 2\pi r^2 + 2\pi rh$ where r is the base radius and h is the height. Solve this equation for h.

6.3. Applications of Linear Equations

KYOTE Standards: *CR 16; CA 14*

Many applied problems that arise in mathematics, science, technology, business and many other fields can be solved by translating them into an algebraic equation and solving the resulting equation. In this section, we examine a variety of such problems that can be solved using a *linear* equation. There are some general guidelines that are helpful in solving *any* of these problems that we discuss below.

Five Steps to Use as Guidelines in Solving Applied Problems with Equations

1. **Define the Variable.** Read the problem carefully. The problem asks you to find some quantity or quantities. Choose one of these quantities as your variable and denote it by a letter, often the letter x. Write out a clear description of what the quantity x represents.
2. **Express All Other Unknown Quantities in Terms of the Variable.** Read the problem again. There are generally unknown quantities in the problem other than the one represented by the variable, say x. Express these unknown quantities in terms of x.
3. **Set up the Equation.** Set up a linear equation that gives a relationship between the variable and the unknown quantities identified in Step 2.
4. **Solve the Equation.** Solve the linear equation you obtain.
5. **Interpret Your Answer.** Write a sentence that answers the question posed in the problem. *Caution*. The variable name x or other unknown quantities expressed in terms of x should not appear in your interpretation.

Example 1. A Coach handbag is on sale for \$247.50 and is advertised as being 25% off its original price. What was its original price?

Solution. We are asked to find the original price. So we define

x =original price of handbag, in dollars　　Step 1

There is no other unknown quantity in this problem, so we can skip Step 2. We take 25% of the original price, or $0.25x$, subtract it from the original price x, and set it equal to the sale price to obtain a relationship between the original price and the sale price

$$x-0.25x=247.50$$ Step 3

We solve this linear equation. Step 4

$x-0.25x=247.50$　　Given equation

$(1-0.25)x=247.50$　　Collect like terms

$0.75x = 247.50$	Simplify
$0.75x = \frac{247.50}{0.75}$	Divide by 0.75
$x = 330$	Simplify

We interpret the answer.

The original price of the handbag is $330.

Example 2. A man weighs 45% more than his daughter. The sum of their weights is 294 pounds. How much does his daughter weigh?

Solution. We are asked to find the daughter's weight. So it makes sense to let

x = daughter's weight in pounds — Step 1

The other unknown quantity is the man's weight. The most difficult part of this problem is to express the man's weight in terms of x, his daughter's weight. The man's weight is equal to his daughter's weight plus 45% more. We know that 45% of x can be written $0.45x$. Thus we can write

$x + 0.45x$=man's weight in pounds — Step 2

The sum of their weights is 294 pounds. This gives us the relationship between these unknown quantities that provides the equation we are seeking:

$x + (x + 0.45x) = 294$ — Step 3

We solve the linear equation. — Step 4

$x + x + 0.45x = 294$	Given equation
$(1 + 1 + 0.45)x = 294$	Collect like terms
$2.45x = 294$	Simplify
$x = \frac{294}{2.45}$	Divide by 2.45
$x = 120$	Simplify

We interpret the answer. — Step 5

The man's daughter weighs 120 pounds.

One approach that may help in setting up the equation is to assign a number to the variable and work out the problem based on this number. The table shows how this could be done for three choices for the daughter's weight.

Daughter's Weight	Man's Weight	Sum
100	100+(.45)(100)=145	245
120	120+(.45)(120)=174	294
140	140+(.45)(140)=203	343

We see how the man's weight is computed in terms of the daughter's weight and how the two are added together to give the sum of their weights. In this case, we happened to get the correct answer by "guessing". This process of "guessing" shows understanding of the problem, but should not be used as a substitute for setting up and solving an appropriate equation. In general, there are just too many possible choices to try them all!

Example 3. Brianna invests a $50,000 inheritance she received from a wealthy aunt in two certificates of deposit. The first paid 3% and the second paid $3\frac{1}{2}\%$ simple interest annually. If her total interest earned from these investments is $1687.50 per year, how much money is invested at each rate?

Solution. There are two unknown quantities in the problem: the amount invested in the first certificate and the amount invested in the second certificate. We can let the variable equal either amount. We arbitrarily let

x=amount invested in the first certificate Step 1

How do we express the amount invested in the second certificate in terms of x? Since the sum of the amounts in both accounts equals 50000, we let

$50000-x$=amount invested in second certificate Step 2

The annual interest earned on the amount invested in the first certificate is $0.03x$ and the second certificate is $0.035(50000-x)$. We add the interest earned in each account and set their sum equal to the total interest earned to obtain an equation relating the two unknown quantities.

Interest earned at 3% + Interest earned at $3\frac{1}{2}\%$ = Total Interest

The equation we seek is therefore

$0.03x+0.035(50000-x)=1687.50$ Step 3

We solve the linear equation. Step 4

$0.03x+0.035(50000-x)=1687.50$	Given equation
$0.03x+1750-0.035x=1687.50$	Expand
$(0.03-0.035)x=1687.50-1750$	Collect like terms
$-0.005x=-62.50$	Simplify
$x=12500$	Divide by -0.005

Since $x=12500$ is the amount invested in the first certificate, the amount invested in the second certificate is $50000-x=50000-12500=37500$.

We can then interpret our answer. Step 5

Brianna invested $12,500 in the first certificate earning 3% *annual interest and $37,500 in the second certificate earning* $3\frac{1}{2}\%$ *annual interest.*

A table assigning a number to the variable and working out the problem based on this number can be useful in helping to set up this problem.

	First Certificate	**Second Certificate**	**Total**
Amount Invested	20000	30000	50000
Interest Earned	$(0.03)(20000)=600$	$(0.035)(30000)=1050$	1650

The total interest earned is $1,650 when $20,000 is invested in the first certificate. This is less than the $1,687.50 required, and so we need to invest less in the first certificate yielding 3% annual interest and more in the second certificate yielding $3\frac{1}{2}\%$ annual interest.

Example 4. A rectangular garden has length 3 feet longer than its width and is enclosed by a fence costing $7.50 per foot. If the entire fence costs $345, what is the length and width of the garden?

Solution. The length and width of the garden are the two unknown quantities we are asked to find. We let

x =width of the garden, in feet Step 1

Since the length of the garden is 3 feet more than its width, we can write

$x+3$=length of the garden, in feet Step 2

The total length of the fence, the perimeter of the rectangle, is another quantity we need to find so that we can use the information in the problem to solve it. We have

$2x+2(x+3)$=length of fence, in feet Step 2

Since the fence costs \$7.50 per foot, has length $2x+2(x+3)$ feet, and costs \$345 altogether, we can form the equation relating the unknown quantities.

$7.50\left(2x+2(x+3)\right)=345$ Step 3

We solve this equation. Step 4

$7.50\left(2x+2(x+3)\right)=345$	Given equation
$7.50\left(4x+6\right)=345$	Collect like terms
$30x+45=345$	Expand
$30x=300$	Subtract 45
$x=10$	Divide by 30

Since $x=10$ is the width of the garden, $x+3=13$ is the length of the garden.

We can then interpret our answer. Step 5

The length of the garden is 13 feet and the width of the garden is 10 feet.

Example 5. A van with no license plate is going 60 miles per hour and passes a truck stop along an interstate highway. A police car passes the truck stop 2 minutes later going 80 miles per hour in hot pursuit of the van. How long will it take for the police car to overtake the van?

Solution. We let

t =time required for police car to overtake the van, in hours Step 1

The distance traveled by the van and the police car from the truck stop to the point where the police car overtakes the van will be the same. But the *time* traveled by the van will be 2 minutes greater than the *time* traveled by the police car. Thus we need to convert 2 minutes to hours to find the time traveled by the van, in hours. We have

$$2\text{ min}\times\frac{1\,hour}{60\text{ min}}=\frac{1}{30}\,hour$$

$t+\dfrac{1}{30}$=time traveled by the van until overtaken by the police car Step 2

The distance traveled by the police car is found by using the familiar formula $d=rt$, where d is the distance traveled, r is the rate and t is the time.

$$80\,\frac{miles}{hour} \times t\ hours = 80t\ miles$$

The distance traveled by the van is

$$60\,\frac{miles}{hour} \times \left(t + \frac{1}{30}\right) hours = 60\left(t + \frac{1}{30}\right) miles$$

We equate these distances to find our equation.

$$80t = 60\left(t + \frac{1}{30}\right)$$ Step 3

We solve this equation. Step 4

$80t = 60\left(t + \frac{1}{30}\right)$	Given equation
$80t = 60t + 2$	Expand
$20t = 2$	Subtract $60t$
$t = \frac{1}{10}$	Divide by 20

To interpret our answer, we might want to convert $\frac{1}{10}$ hour to minutes.

$$\frac{1}{10} hour \times \frac{60\min}{hour} = 6\min$$

The police car overtakes the van in 6 minutes. Step 5

Example 6. A coin purse contains nickels, dimes and quarters worth $5.80. There are one half as many quarters as dimes and one third as many nickels as dimes in the purse. How many coins of each type are in the purse?

Solution. We let

x =number of dimes in the purse Step 1

We can then find expressions for the number of quarters and nickels in term of x.

$\frac{1}{2}x$ =number of quarters in the purse

$\frac{1}{3}x$=number of nickels in the purse Step 2

We then find the value, in cents, of each of the dimes, quarters and nickels and set this equal the total value of all the coins, 580, in cents. We obtain

$$10x+25\left(\frac{1}{2}x\right)+5\left(\frac{1}{3}x\right)=580$$ Step 3

We solve this equation. Step 4

$10x+25\left(\frac{1}{2}x\right)+5\left(\frac{1}{3}x\right)=580$	Given equation
$6\left(10x+25\left(\frac{1}{2}x\right)+5\left(\frac{1}{3}x\right)\right)=6(580)$	Multiply by 6
$60x+75x+10x=3480$	Expand
$145x=3480$	Collect like terms
$x=24$	Divide by 145

Since $x=24$, $\frac{1}{2}x=\frac{1}{2}(24)=12$ and $\frac{1}{3}x=\frac{1}{3}(24)=8$. We can then interpret our answer.

Step 5

There are 24 dimes, 12 quarters and 8 nickels in the coin purse.

Example 7. A manufacturer has 300 gallons of an orange soda drink containing 10% pure orange juice. She wants to mix the 300 gallons with pure orange juice to produce a mixture containing 25% pure orange juice. How many gallons of pure orange juice does she need to add?

***Solution*.** We let

x=gallons of pure orange juice to be added Step 1

The mixture of 300 gallons of orange soda and the x gallons of pure orange juice will contain $300+x$ gallons, 25% of which will be pure orange juice. Thus the number of gallons of pure orange juice in the mixture is given by

$$0.25(300+x)$$ Step 2

The number of gallons of pure orange juice can also be written as the sum of the gallons pure orange juice in the 300 gallons of orange soda and the x gallons of pure orange juice *before* they are mixed.

$$0.10(300)+x$$ Step 2

The equation we seek is therefore

$$0.10(300)+x=0.25(300+x)$$ Step 3

We solve the equation. Step 4

$0.10(300)+x=0.25(300+x)$	Given equation
$30+x=75+0.25x$	Expand
$x-0.25x=75-30$	Subtract $0.25x$ and 30
$0.75x=45$	Simplify
$x=60$	Divide by 0.75

We interpret out answer. Step 5

She must add 60 gallons of pure orange juice.

Exercise Set 6.3

1. The length of a rectangle is 2 feet longer than its width. If its width is x feet, find its area and its perimeter in terms of x.

2. The width of a rectangle is half its length. If its length is x feet, find its area and its perimeter in terms of x.

3. The length of a rectangle is 3 feet longer than twice its width. If its width is x feet, find its area and its perimeter in terms of x.

4. If the length of a rectangle is l meters and its area is A square meters, find its width in terms of l and A.

5. If the length of a rectangle is l meters and its perimeter is P meters, find its width in terms of l and P.

6. The height of a triangle is 3 times longer than its base. If its base is b inches, find its area in terms of b.

7. If the height of a triangle is h centimeters and it area is A square centimeters, find its base b in terms of h and A.

8. The average of three numbers x, y and z is A. Find z in terms of A, x and y.

9. (a) If x is the smallest of three consecutive odd integers, what is the sum S of these three integers in terms of x? **(b)** If x is between the smallest and largest of three consecutive odd integers, what is the sum S of these three integers in terms of x?

10. A Corvette going east at x miles per hour and a Mustang going west at 5 miles per hour faster than the Corvette pass each other on a long straight highway. How many miles apart are the two cars 10 minutes after passing each other, in terms of x?

11. Chuck goes on a trip in his Prius hybrid. He goes x miles and averages 50 miles per gallon of gas. If gas costs $3.50 per gallon, how much did he pay for gas on this trip in terms of x?

12. Hannah earns $12 per hour at her job, but earns one and a half times that rate for overtime hours in excess of 35 hours per week. How much does she earn one week if works x overtime hours, in terms of x?

13. A master carpenter makes twice as much per hour as his assistant. On one job, the assistant works 5 hours more than the carpenter. If the assistant makes x dollars per hour and works h hours on this job, how much altogether are they paid for the job, in terms of x and h?

14. Beth works out by running x miles from her house each day at the rate of 8 miles per hour and walking back home at the rate of 3 miles per hour. How many hours does it take her to complete this workout in terms of x?

15. A woman earning a salary of x dollars per year gets a $3\frac{1}{2}\%$ raise. **(a)** What is her raise in terms of x? **(b)** What is her new annual salary in terms of x?

16. A shirt selling for x dollars is marked down 15%. What is the new price of the shirt in terms of x?

17. A man invests x dollars in an account paying 4% annual interest and twice as much in an account earning 6% interest. **(a)** How much does the man invest in these two accounts altogether in terms of x? **(b)** How much annual interest does the man earn from these accounts in terms of x?

18. Janis invests $4,000 dollars, part in a savings account earning 2.5% annual interest and the rest in a certificate of deposit earning 4.5% annual interest. If she invested x dollars in the savings account, how much interest did she earn on her investment in terms of x?

19. A coin purse contains nickels, dimes and quarters. It contains x quarters, twice as many dimes as quarters, and 3 more nickels than quarters. **(a)** How many coins does the purse contain in terms of x? **(b)** How much money, in cents, does the purse contain in terms of x?

20. Chris has only \$5, \$10, and \$20 bills in his wallet. It contains x \$10 bills, 4 more than twice as many \$20 bills as \$10 bills and 7 more \$5 bills as \$10 bills. **(a)** How many bills does are in his wallet in terms of x? **(b)** How much money, in dollars, is in the wallet in terms of x?

21. Select Blend coffee contains 35% Columbian coffee and Premium Blend coffee contains 75% Columbian coffee. If x pounds of Brand A coffee is blended with y pounds of Brand B coffee, how many pounds of Columbian coffee is in this mixture, in terms of x and y?

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22. The sum of four consecutive even integers is 140. What are the four integers?

23. A company president makes 80% more than her vice president. Their combined annual salary is \$630,000. What is the annual salary of these two executives?

24. A car dealer reduces the price of a Ford Focus by 20%. What was the price of the car before this reduction if the current price is \$14, 560?

25. In 2011, Jay Bruce of the Cincinnati Reds hit 3 more home runs than his teammate Joey Votto. Together they hit 61 home runs. How many home runs did each player hit?

26. In the 2011-2012 University of Kentucky national championship season, freshman stars Anthony Davis, Michael Kidd-Gilchrist and Terrence Jones combined to score 1,512 points. If Kidd-Gilchrist scored 7 more points than Jones and Davis scored 91 more points than Kidd-Gilchrist, how many points did each player score?

27. A student has scores of 84 and 92 on the first two tests in her Algebra II class. What score does she need to get on the third test to make the average on the three tests 90 so that she can get an A in the course?

28. The grade in a geometry class consists of 100 points for homework, 100 points each for two exams and 200 points for the final exam. Students who earn 90% or more of the total points possible earn an A. Gail has 95 points on homework, 82 points on the first exam and 88 points on the second exam. What is the minimum score on the final exam that she needs to earn an A?

29. A rectangular garden has length 35 feet and surrounded by 88 feet of fencing. What is the area of the garden?

30. A rectangle is 4 inches longer than it is wide and has a perimeter of 160 inches. What are the length and width of the rectangle?

31. A rectangle is twice as long as it is wide and has a perimeter of 300 feet. What is the area of the rectangle?

32. A rectangular field with length 28 feet and area 672 square feet is to be enclosed by a fence. How much will it cost to purchase the fence if it sells for $3.50 per linear foot?

33. A man is four times older than his daughter. The sum of their ages is 45. How old is his daughter?

34. A woman is twice as old as her son and 12 years ago, she was three times older than her son. How old is her son?

35. Joanne invests a total of $7,000 in two accounts, the first paying a simple annual interest rate of 4% and the second a simple annual interest rate of 6%. After one year the total interest earned on this investment is $375. How much did she invest in each account?

36. A coin purse contains nickels, dimes and quarters worth $4.35. There are twice as many nickels as dimes and 3 more quarters than dimes. How many coins of each type are in the purse?

37. An envelope contains $1, $5, $10 and $20 bills worth $210. There are 12 more $1 bills as $5 bills, twice as many $10 bills as $5 bills, and an equal number of $10 and $20 bills. How many bills of each type are in the envelope?

38. Financial consultant E. Z. Money has a client who has invested $40,000 in an account yielding 4% annual interest. She wants her client to invest in another account yielding 7% annual interest to ensure that the annual interest earned from the two accounts combined is 5%. How much should be invested in the account yielding 7% annual interest?

39. Romeo and Juliet have a lover's quarrel. They stomp off in opposite directions with Juliet walking 2 feet per second faster than Romeo. At what rate, in feet per second, is each of them walking if they are 480 feet apart after one minute?

40. Sleazy, the little-known eighth dwarf in the classic Snow White story, is appropriately named. When Snow White walked into the woods one day at 4 feet per second, Sleazy followed her half a minute later walking briskly at 6 feet per second. How long will it take for Sleazy to catch up with Snow White?

41. Speedy is another dwarf who hangs out with Sleazy. He stays fast by running at 7 miles per hour from his tiny cottage and then walking back along the same path at 3 miles per hour. How many miles does he travel if this workout takes 1 hour to complete?

42. Two cars are 30 miles apart and moving towards each other on the same highway. One car is going 10 miles per hour faster than the other. How fast is each of the cars going if they pass each other 15 minutes later?

43. Curly, Larry and Moe all work for the same company and they compare their annual salaries. Curly makes 20% more than Moe and Larry makes 30% more than Moe. Altogether they make $183,750 per year. How much per year does each man make?

44. Chemistry professor M. T. Beaker wants to mix a solution containing 60% acid with a solution containing 30% acid to produce a 300-milliliter mixture containing 50% acid. How many milliliters of each solution should he use?

45. Professor Beaker has a 50-milliliter solution containing 60% acid. How many milliliters of pure water should he add to this solution to produce a solution containing 40% acid?

46. Select Blend coffee contains 35% Columbian coffee and Premium Blend coffee contains 75% Columbian coffee. A store manager wants to mix these two brands of coffee to obtain a 30-pound mixture containing 50% Columbian coffee. How many pounds of each brand should he use?

47. Thrifty Blend coffee does not contain any Columbian coffee and Premium Blend coffee contains 75% Columbian coffee. A store manager wants to mix 20 pounds of Premium coffee with some Thrifty Blend coffee to obtain mixture containing 50% Columbian coffee. How many pounds of Thrifty Blend should he use?

48. Thrifty Blend coffee sells for $2.00 a pound and Premium Blend coffee sells for 3.50 a pound. A store manager wants to mix these two blends to produce a 30-pound mixture that sells for $2.50 a pound. How many pounds of each blend should she mix?

49. A master carpenter makes $55 an hour and his assistant makes $30 an hour. On one job, the assistant works 5 hours longer than the carpenter and together they earn $2,020 in wages. How many hours did each man work?

50. A married couple takes their three children to a ball game. A ticket for a child costs 40% less than a than a ticket for an adult. If the cost of admission for the family is $133, what is the cost of a ticket for an adult?

6.4. Solve Linear Inequalities in one Variable

KYOTE Standards: *CR 17; CA 10*

Inequalities

An inequality is a statement that one mathematical expression is either less than (<), greater than (>), less than or equal to ($\leq$) or greater than or equal to ($\geq$) another mathematical expression. The statement $3 \leq 7-2$ is an inequality.

In this section, we discuss inequalities that involve a variable such as

$$2x-3 \leq 9$$

We look for values of x that make the inequality $2x-3 \leq 9$ a true statement. The following table gives some values of x that *satisfy* the inequality (make it a true statement) and some values of x that do not satisfy the inequality. A value of x that satisfies the inequality is called a *solution* to the inequality. The collection of all solutions is called the *solution set* of the inequality.

x	$2x-3 \leq 9$	**Solution?**
4	$2\cdot 4-3=5 \leq 9?$	Yes
5	$2\cdot 5-3=7 \leq 9?$	Yes
6	$2\cdot 6-3=9 \leq 9?$	Yes
7	$2\cdot 7-3=11 \leq 9?$	No
8	$2\cdot 8-3=13 \leq 9?$	No

We see from the table that 4, 5 and 6 are solutions of the inequality and are in its solution set whereas 7 and 8 are not solutions and are not in its solution set.

Two linear inequalities are *equivalent* if they have exactly the same solution set just as two linear equations are equivalent if the have exactly the same solution.

We can use algebra to solve a linear inequality by applying the properties of inequality listed in the table below in the same way we used algebra to solve linear equations using the properties of equality. The idea is to reduce the linear inequality to simpler, equivalent inequalities until the solution is obtained.

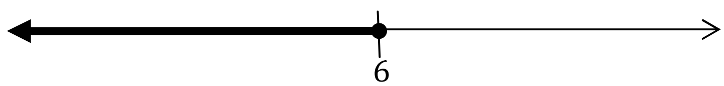
Properties of Inequality

1. Adding (or subtracting) the same number to both sides of an inequality gives an equivalent inequality. **2.** Multiplying (or dividing) both sides of an inequality by the same *positive* number gives an equivalent inequality. **3.** Multiplying (or dividing) both sides of an inequality by the same *negative* number gives an equivalent inequality with the *direction of the inequality reversed*.

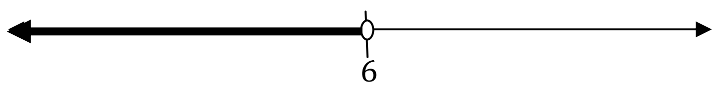
Suppose we would like to solve our example inequality $2x-3\le 9$ above using algebra. We obtain the following sequence of equivalent inequalities by applying the properties of inequality.

$2x-3\le 9$	Given inequality
$2x\le 12$	Add 3; property 1
$x\le 6$	Divide by 2; property 2

The solution set consisting of all real numbers x such that $x\le 6$ is reinforced by the solutions we obtained in the table above. The *graph* of this solution set is the bold portion of the line on the number line shown.

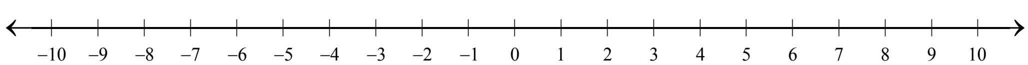


The *solid dot* indicates that the number 6 is a solution and should be included in the solution set. The *open circle* on the number line graph below, however, indicates that 6 is *not* in the solution set of the inequality $x<6$.



Property 3

The third property of inequality listed above is different from any of the properties of equality and deserves some discussion: Multiplying (or dividing) both sides of an inequality by the same *negative* number gives an equivalent inequality with the *direction of the inequality reversed*. It is a direct consequence of how numbers are ordered on the number line.



For example, $2<5$ because 2 is to the *left* of 5 on the number line. On the other hand, $-2>-5$ since -2 is to the *right* of -5 on the number line. Thus multiplying or dividing both sides of $2<5$ by -1 reverses the direction of the inequality and we obtain $-2>-5$. The same thing happens if $2<5$ is multiplied or divided by *any* negative number. Multiplying both sides of $2<5$ by -3, for instance, reverses the direction of the inequality and gives $-6>-15$, a true statement since -6 is to the right of -15 on the number line.

This same argument works for inequalities such as $-7 < -3$ (both numbers are negative) and $-4 < 6$ (one number is negative and the other is positive). You might want to try it and see for yourself!

Interval Notation

Certain sets of real numbers, including solution sets of inequalities, correspond to line segments on the number line and are called *intervals*. Suppose $a < b$ are two numbers on the number line. The interval (a,b), where a and b are enclosed by parentheses, consists of all numbers between a and b but not the endpoints of the interval a and b. The interval $[a,b]$, where a and b are enclosed by square brackets, consists of all numbers between a and b including a and b.

A right endpoint of infinity (∞) indicates that the interval has no right endpoint but extends infinitely far to the right on the number line. Thus the interval (a,∞) consists of all real numbers greater than a. The interval $[a,\infty)$ consists of all real numbers greater than or equal to a. The symbol ∞ is not a number and so is not enclosed by a square bracket on the right.

A left endpoint of minus infinity ($-\infty$) indicates that the interval has no left endpoint but extends infinitely far to the left on the number line. Thus the interval $(-\infty,b)$ consists of all real numbers less than b. The interval $(-\infty,b]$ consists of all real numbers less than or equal to b. The symbol $-\infty$ is not a number and so is not enclosed by a square bracket on the left.

Example 1. Solve the linear inequality. Graph its solution set and express it in interval notation.

(a) $-2x-3<-8$ **(b)** $2x-7 \ge 5x+4$

Solution. **(a)** We apply the properties of inequality to get all terms involving x on one side of the inequality and all terms that are numbers alone on the other side just as we did when solving a linear equation. Finally, we divide by the coefficient of x in the resulting inequality to obtain the solution. The steps are as follows:

$-2x-3<-8$ Given inequality
$-2x<-5$ Add 3
$x>\frac{5}{2}$ Divide by -2; reverse direction

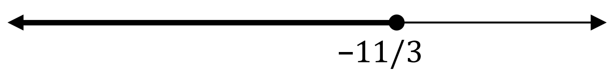
The graph of the solution set is the bold portion of the line shown:

It can be expressed in interval notation as $\left(\frac{5}{2},\infty\right)$.

(b) We proceed as in part (a), but we bring all terms involving x to the right side of the inequality to avoid reversing the direction of the inequality and all bring all the numbers to the left. We obtain

$2x-7\geq 5x+4$	Given equation
$-11\geq 3x$	Subtract $2x$; subtract 4
$-\frac{11}{3}\geq x$	Divide by 3; do *not* reverse direction

The graph of the solution set is the bold portion of the line shown:



It can be expressed in interval notation as $\left(-\infty,-\frac{11}{3}\right]$.

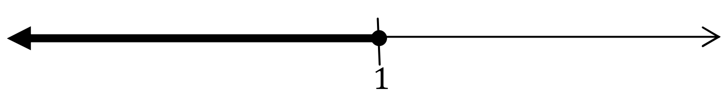
Example 2. Solve the linear inequality. Graph its solution set and express it in interval notation.

(a) $2(3x-2)\leq 5-(4-x)$ **(b)** $\frac{1}{2}x-\frac{2}{3}x>\frac{1}{3}+x$

Solution. We first expand to remove the parentheses. We then apply the properties of inequality to get all terms involving x on one side and all terms involving numbers alone on the other side. Finally, we divide by the coefficient of x in the resulting inequality to obtain the solution. The steps are as follows:

$2(3x-2)\leq 5-(4-x)$	Given inequality
$6x-4\leq 5-4+x$	Expand
$5x\leq 5$	Subtract x; add 4
$x\leq 1$	Divide by 5

The graph of the solution set is the bold portion of the line shown:

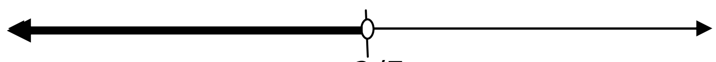


It can be expressed in interval notation as $(-\infty,1]$.

(b) We multiply both sides of the inequality by the *LCD* of the denominators of the fractions involved to clear fractions just as we do for equations. We then solve the resulting inequality as in part (a). We obtain

$\frac{1}{2}x-\frac{2}{3}x>\frac{1}{3}+x$	Given inequality
$6\left(\frac{1}{2}x-\frac{2}{3}x\right)>6\left(\frac{1}{3}+x\right)$	Multiply by 6
$3x-4x>2+6x$	Expand
$-7x>2$	Subtract $6x$; collect like terms
$x<-\frac{2}{7}$	Divide by -7; reverse direction

The graph of the solution set is the bold portion of the line shown:



It can be expressed in interval notation as $\left(-\infty,-\frac{2}{7}\right)$.

Exercise Set 6.4

Solve the linear inequality. Graph its solution set and express it in interval notation.

1. $3x-7\geq 11$

2. $1-2x<5$

3. $19-4x<0$

4. $5x-1\leq 2x-7$

5. $4x+12\geq 9x+8$

6. $-(2x-3)\leq 0$

7. $-3x\geq -7$

8. $3x+11\leq 6x+8$

9. $-7x-3x+23>3$

10. $\frac{3}{4}x\geq 1$

11. $\frac{x}{7}-5<2$

12. $0.2(x-3)<4$

13. $\frac{2x}{3} < 3.8$

14. $2-(1-3x) \geq 5-2(3-x)$

15. $\frac{1}{2}\left(x-\frac{1}{3}\right) > 2$

16. $7x+1 \leq 3-(2x-4)$

17. $\frac{3}{4}x-\frac{2}{5}x > 1$

18. $2(1-3x) \leq 3(2+x)$

19. $0.01x+2.6 \geq 1.3$

20. $-\frac{5}{3}x-18 > x+\frac{1}{3}x$

21. $\frac{1}{3}x-\frac{1}{4}x \leq 5-x$

22. $\frac{5x}{12} \geq \frac{7x}{18}+\frac{1}{6}$

Chapter 7. Lines

7.1. Slopes and Graphs of Lines

KYOTE Standards: *CR 18; CA 16*

We know from high school geometry that there is exactly one line that passes through (or contains) two given points. In this section we define the *slope* of a line. We show how the slope of a line can be found given any two points on the line.

We show that there is exactly one nonvertical line with a given slope that contains a given point. We show how the graph of the line can be constructed using this slope and this point. We leave a discussion of equations of lines for the next section.

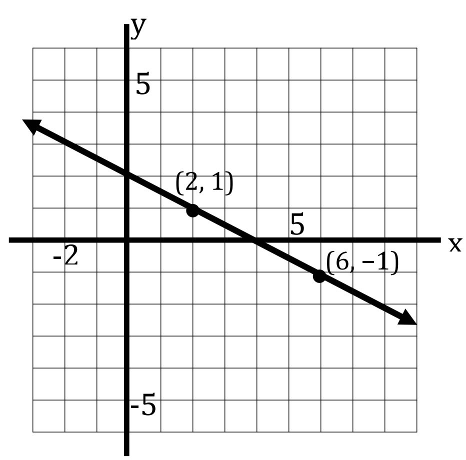
Definition 1. The *slope* of the nonvertical line containing two points P and Q in the xy-plane is

$$slope = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x}$$

The *rise* is the change in the y-coordinate in going from P to Q and the *run* is the change in the x-coordinate in going from P to Q.

Example 1. Find the slope of the line passing through $(2,1)$ and $(6,-1)$.

Solution. These two points determine a unique line and it is a good idea to graph this line to help visualize and calculate its slope.



If we go from $(2,1)$ to $(6,-1)$, we see that the x-coordinate *increases* by 4 and the y-coordinate *decreases* by 2. Thus the slope of the line is

$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{-2}{4} = -\frac{1}{2}$$

If we go in the opposite direction from $(6,-1)$ to $(2,1)$, we see that the x-coordinate *decreases* by 4 and the y-coordinate *increases* by 2. Thus the slope of the line is

$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{2}{-4} = -\frac{1}{2}$$

We see that the slope is the same regardless of the direction we choose.

Example 2. A line passes through the point $(1,3)$ and has slope 2. Construct a table of at least four points on the line and sketch its graph.

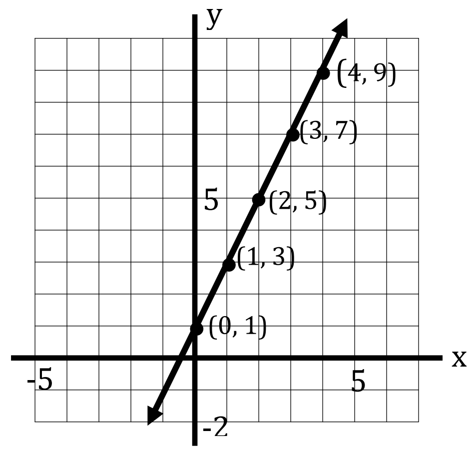
Solution. We start at the point $(1,3)$. If we *increase* the x-coordinate by 1 and *increase* the y-coordinate by 2, we arrive at another point $(2,5)$ on the line since the slope of the line is 2. If we *increase* the x-coordinate by 2 and *increase* the y-coordinate by $2 \cdot 2 = 4$, we arrive at yet another point $(3,7)$ on the line for the same reason. If we *increase* the x-coordinate by 3 and *increase* the y-coordinate by $2 \cdot 3 = 6$, we arrive at still another point $(4,9)$ on the line.

On the other hand, if we start at $(1,3)$, *decrease* the x-coordinate by 1 and *decrease* the y-coordinate by 2, we arrive at another point $(0,1)$ on the line. The change -2 in y divided by the change -1 in x is equal to 2, the slope of the line and confirms that $(0,1)$ is on the line.

We can generate as many points on the line as we want using this reasoning. The table below shows the four points we generated in this way.

x	y
1	3
2	5
3	7
4	9
0	1

The graph of the line can then be easily sketched. The coordinates of the points in the table are included on the graph.



Notice that regardless of the pair of points we choose on the line, the change in the y-coordinate divided by the change in the x-coordinate in going from one point to the other is always equal to the slope of the line. For example, if we go from $(0,1)$ to $(3,7)$, the change 6 in the y-coordinate divided by the change 3 in the x-coordinate is always equal to 2.

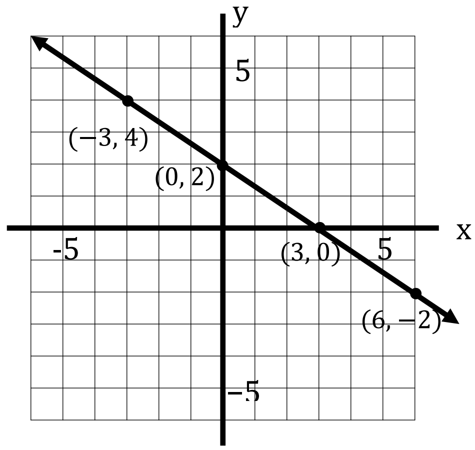
Example 3. Find the slope of the line through $(0,2)$ and $(3,0)$. Find two additional points on the line. Sketch the graph and place the coordinates of all four points on the graph.

Solution. If we go from $(0,2)$ to $(3,0)$, we see that the x-coordinate *increases* by 3 and the y-coordinate *decreases* by 2. Thus the slope of the line is

$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{-2}{3} = -\frac{2}{3}$$

We find two additional points on the line using the same approach we used in Example 2. We start at $(3,0)$. We *increase* the x-coordinate by 3 and *decrease* the y-coordinate by 2 to arrive at another point $(6,-2)$ on the line. To find another point, we also choose to start at $(3,0)$. (Note that we could also start at any other point on the line, $(0,2)$ or $(6,-2)$, for example.) If we *decrease* the x-coordinate by 6 and *increase* the y-coordinate by 4, then we arrive at yet another point $(-3,4)$ on the line.

We graph the line shown below and we place the points $(0,2)$, $(3,0)$, $(6,-2)$ and $(-3,4)$ on the graph.



Horizontal and Vertical Lines

Horizontal and vertical lines are special types of lines that we consider in Example 4. As we shall see, horizontal lines have slope 0 and vertical lines have no slope.

Example 4. Find the slope of the line (if it exists) through the given pair of points and sketch its graph.

(a) $(-1,3)$, $(4,3)$ **(b)** $(2,-2)$, $(2,5)$

Solution. **(a)** If we go from $(-1,3)$ to $(4,3)$, we see that the x-coordinate *increases* by 5 and the change in the y-coordinate is 0. Thus the slope of the line is

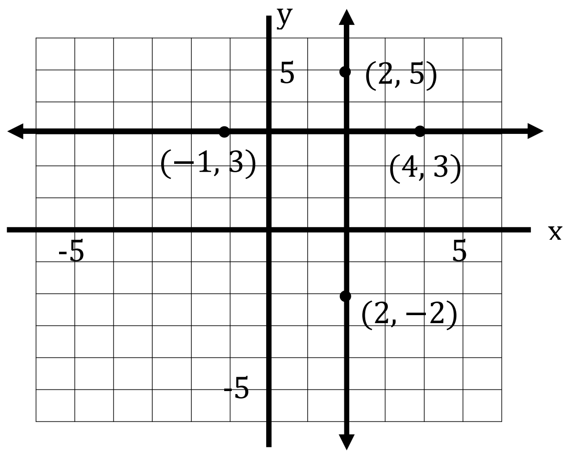
$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{0}{5} = 0$$

Any other point whose y-coordinate is 3 is also on this line. For example, $(7,3)$ is on the line since the change in y going from $(4,3)$ [or $(-1,3)$] to $(7,3)$ equals 0. Thus this line is a horizontal line and its graph is the horizontal line shown below.

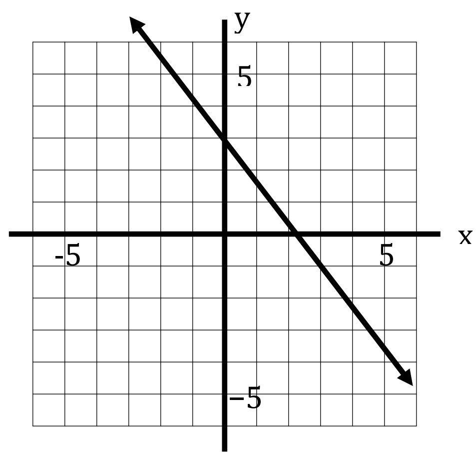
(b) If we go from $(2,-2)$ to $(2,5)$, we see that the change in the x-coordinate is 0 and the y-coordinate increases by 7. Since we cannot divide by 0, the ratio

$$\frac{change\ in\ y}{change\ in\ x}$$

is not defined and the line has no slope. Any other point whose x-coordinate is 2 is also on this line. For example, $(2,1)$ is on this line since the change in x going from $(2,5)$ to $(2,1)$ equals 0. Thus this line is a vertical line and its graph is the vertical line shown below.



Example 5. Find the slope of the line whose graph is shown.



Solution. Points with integer coordinates occur at the intersection of two grid lines and so we can identify their coordinates exactly. Our goal is to choose two such points and calculate the slope of the line using these two points.

We see from the graph that $(0,3)$ and $(3,-1)$ are two such points in this case. If we go from $(0,3)$ to $(3,-1)$, we see that the x-coordinate *increases* by 3 and the y-coordinate *decreases* by 4. Thus the slope of the line is

$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{-4}{3} = -\frac{4}{3}$$

Exercise Set 7.1

Construct a table containing the given point and at least four other points on the line as in Example 2. Sketch its graph.

1. A line of slope 1 passing through the point $(2,1)$.

2. A line of slope -1 passing through the point $(3,1)$.

3. A line of slope -2 passing through the point $(4,-1)$.

4. A line of slope 0 passing through $(-1,-2)$.

5. A line with no slope passing through $(4,5)$.

6. A line of slope 0 passing through $(4,5)$.

7. A line of slope $\frac{1}{2}$ passing through $(2,4)$.

8. A line of slope $-\frac{1}{2}$ passing through $(0,0)$.

9. A line of slope $\frac{2}{3}$ passing through $(0,0)$.

10. A line of slope -1 passing through $(-2,3)$.

11. A line of slope $-\frac{1}{4}$ passing through $(0,1)$.

12. A line of slope 3 passing through $(-1,-3)$.

Find the slope of the line (if it exists) through the given pair of points and sketch its graph. Find two additional points on the line and place the coordinates of all four points on the graph as in Example 3.

13. $(0,0),(2,1)$

14. $(1,2),(2,1)$

15. $(0,0),(3,-1)$

16. $(2,3),(2,5)$

17. $(-1,0),(-1,3)$

18. $(3,1),(6,4)$

19. $(-4, 0), (0, 2)$

20. $(0, 0), (4, 3)$

21. $(3, -5), (3, 2)$

22. $(4, -2), (4, 3)$

23. $(-1, 0), (-3, 0)$

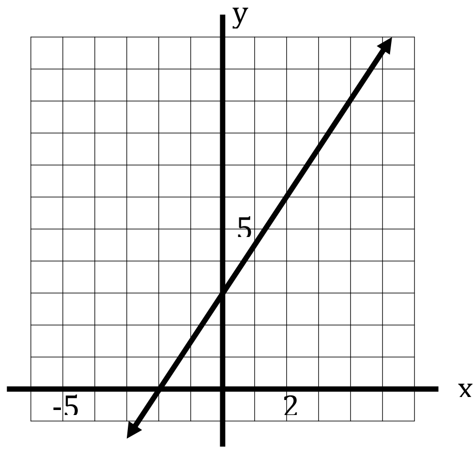
24. $(-2, 3), (3, -2)$

25. $(-3, -1), (1, -2)$

26. $(1, 1), (4, 3)$

Find the slope of the line whose graph is shown.

27.



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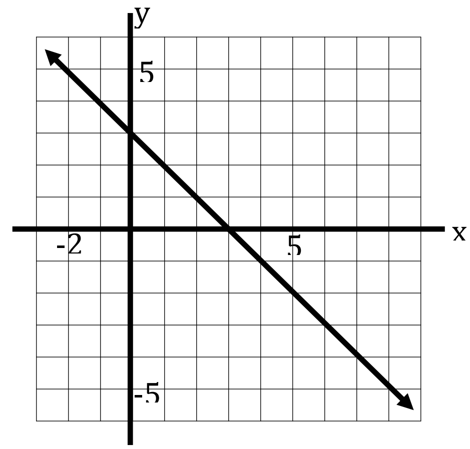

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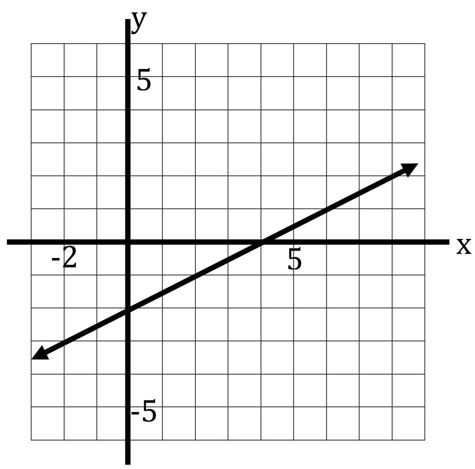

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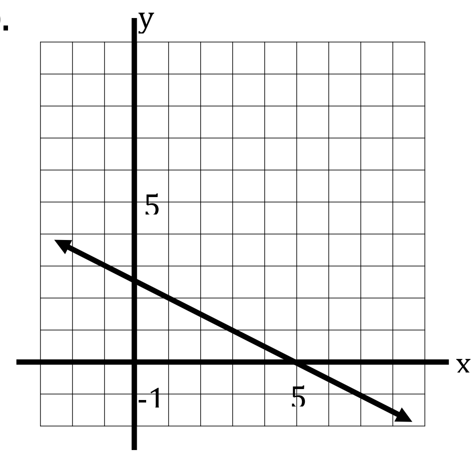

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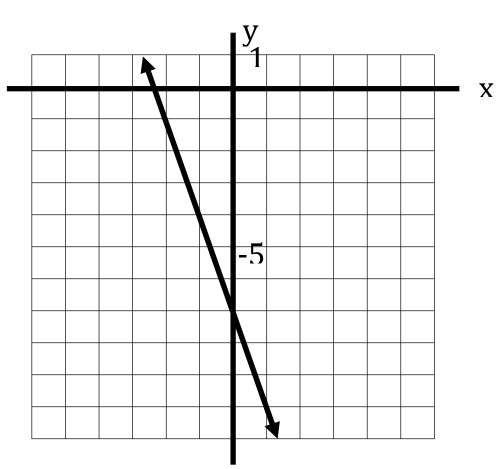

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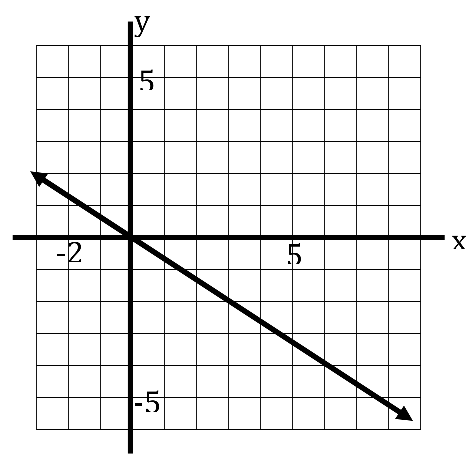


7.2. Equations and Graphs of Lines

KYOTE Standards: *CR 18, CR 19; CA 16*

In the previous section, we discussed lines, their slopes and their graphs, from a purely geometric perspective. In this section, we focus on the profound idea that any line in the plane can be viewed as linear equation in two variables and, conversely, any linear equation in two variables can be viewed a line. The study of lines provides an elegant connection between geometry and algebra.

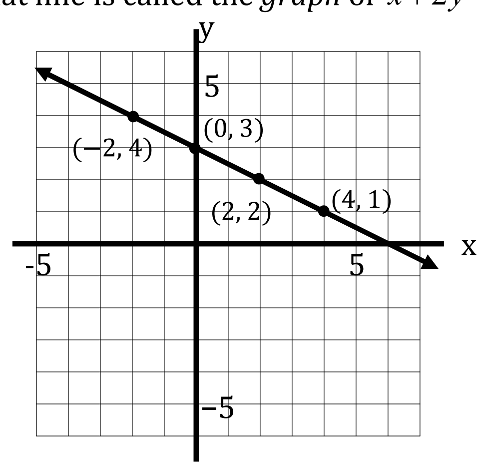
We consider as an example the following linear equation in two variables x and y:

$$x+2y=6$$

A pair of real numbers (x,y) that make this equation a true statement is called a *solution* to the equation. A solution to the equation is said to *satisfy* the equation. For example, the pair $(4,1)$ is a solution to the equation since it makes the equation $4+2(1)=6$ a true statement. The pair $(3,4)$ is *not* a solution since it does not make $3+2(4)=6$ a true statement. The pair $(4,1)$ satisfies the equation $x+2y=6$ but the pair $(3,4)$ does not. We consider some additional pairs in the following table.

x	y	$x+2y$	Solution?
0	3	6	Yes
2	2	6	Yes
–2	4	6	Yes
6	0	6	Yes
4	3	10	No
0	0	0	No
–2	5	8	No

If we plot the five solutions we have obtained and "connect the dots", it does appear that they form a line. In fact, we will show that the set of all solutions to $x+2y=6$ does form a line, and that line is called the *graph* of $x+2y=6$.



The Slope-Intercept Equation of a Line

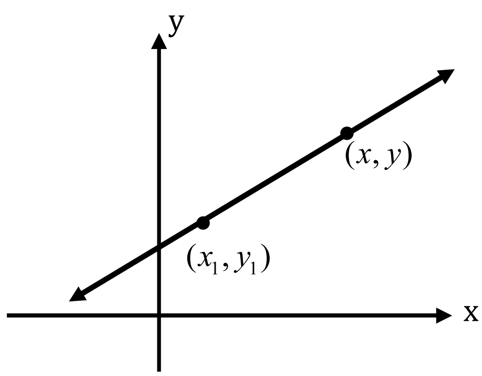
We saw in the previous section how to construct the *graph* of a line given the slope of the line and a point on the line. We now use this information to construct an *equation* of such a line.

Suppose that a line has slope m and passes through the point (x_1, y_1) as shown in the graph below. We can use the property of slope to find an equation of this line. Suppose that (x, y) is *any* other point on the line. If we start at (x_1, y_1) and go to (x, y), then the change in the y-coordinate is $y - y_1$ and the change in the x-coordinate is $x - x_1$. The ratio of these changes must be m, the slope of the line. Thus we have

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y - y_1}{x - x_1} = m$$

We solve the latter equation for y to obtain

$\frac{y - y_1}{x - x_1} = m$	Given equation
$y - y_1 = m(x - x_1)$	Multiply by $x - x_1$
$y = m(x - x_1) + y_1$	Add y_1
$y = mx - mx_1 + y_1$	Expand
$y = mx + b$	Set $b = -mx_1 + y_1$



If we set $x = 0$ in the equation $y = mx + b$, we obtain $y = b$. We call b the y*-intercept* of the line since the point $(0, b)$ is where the line intersects the y-axis.

Therefore we call $y = mx + b$ the *slope-intercept form* of the equation for the line.

The Slope-Intercept Equation of a Line

If a line has slope m and y-intercept b, then an equation of the line given by

$$y = mx + b$$

is said to be in *slope-intercept form*.

Example 1. Find the slope-intercept form of the equation of the line that has slope $-\frac{1}{2}$ and passes through the point $(3,1)$.

Solution. We consider two approaches to solving this problem.

Method 1. We follow the logic in the derivation of the slope-intercept form of the equation discussed above. We let (x,y) be any point on the line other than $(3,1)$. Since the slope of the line is $-\frac{1}{2}$ and (x,y) is on the line, the coordinates x and y satisfy equation

$$\frac{y-1}{x-3} = -\frac{1}{2}$$

We solve this equation for y to obtain

$\frac{y-1}{x-3} = -\frac{1}{2}$	Given equation
$y-1 = -\frac{1}{2}(x-3)$	Multiply by $x-3$
$y-1 = -\frac{1}{2}x + \frac{3}{2}$	Expand
$y = -\frac{1}{2}x + \frac{5}{2}$	Add 1

The equation of the line in slope-intercept form is therefore $y = -\frac{1}{2}x + \frac{5}{2}$.

Method 2. Since the slope of the line is $-\frac{1}{2}$, we know the slope-intercept form of the line is

$y = -\frac{1}{2}x + b$	The variable b is the y-intercept.

The point $(3,1)$ is on the line and thus satisfies this equation $y=-\frac{1}{2}x+b$. We can use this to solve for b.

$1=-\frac{1}{2}(3)+b$ We solve for b to obtain $b=\frac{5}{2}$.

The equation of the line in slope-intercept form is therefore $y=-\frac{1}{2}x+\frac{5}{2}$. This answer is the same as the one obtained using Method 1.

Example 2. Find the equation of the line in slope-intercept form that passes through the points $(-4,2)$ and $(1,-3)$.

Solution. We first find the slope of the line. If we go from $(-4,2)$ to $(1,-3)$, we see that the x-coordinate *increases* by 5 and the y-coordinate *decreases* by 5. Thus the slope of the line is

$$slope=\frac{change\ in\ y}{change\ in\ x}=\frac{-5}{5}=-1$$

The equation of the line has the form

$y=-x+b$ The variable b is the y-intercept.

We can use either of the two points $(-4,2)$ or $(1,-3)$ to find b since both points are on the line and so must satisfy $y=-x+b$. If we use $(-4,2)$, then we obtain

$2=-(-4)+b$ We solve for b to obtain $b=-2$.

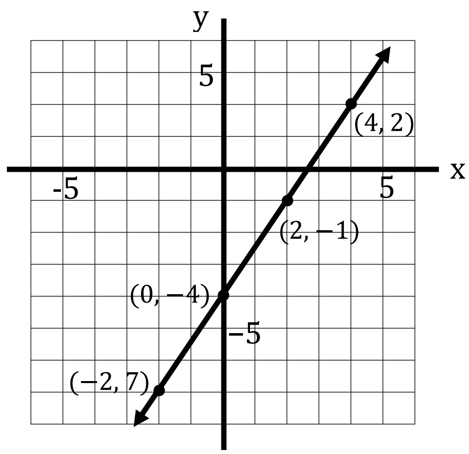
The equation of the line in slope-intercept form is therefore $y=-x-2$.

Example 3. Sketch the graph of the line $y=\frac{3}{2}x-4$. Find at least three points on the line and place the coordinates of these points on your graph.

Solution. We know the slope $\frac{3}{2}$ of the line and a point $(0,-4)$ on the line. So we could use the approach discussed in the previous section to find additional points and then sketch the graph. But since we have an explicit formula for y in terms of x, it is easier to construct a table of points on the line. We choose even integers for x so that the points we choose will have integer coordinates.

x	$y=\frac{3}{2}x-4$
−2	−7
0	−4
2	−1
4	2

The graph is now easily sketched and the four points in the table are placed on the graph.



Definition 1. A *linear equation in two variables* x and y is an equation that can be written in the *general form* $Ax+By=C$, where A, B and C are real numbers and A and B are not both equal to 0.

Every linear equation in general form is a line and we can divide these lines into three distinct categories.

1. **Horizontal Line.** If $A=0$ and $B\neq 0$, then $Ax+By=C$ is a horizontal line with equation $y=\frac{C}{B}$. A horizontal line has slope 0.
2. **Vertical Line.** If $A\neq 0$ and $B=0$, then $Ax+By=C$ is a vertical line with equation $x=\frac{C}{A}$. A vertical line has no slope.
3. **Slanted Line.** If $A\neq 0$ and $B\neq 0$, then $Ax+By=C$ can be put in slope-intercept form as $y=-\frac{A}{B}x+\frac{C}{B}$, where $-\frac{A}{B}$ is the slope and $\frac{C}{B}$ is the y-intercept. A slanted line has a nonzero slope.

Intercepts of a Line

1. The y ***-intercept*** of the line $Ax+By=C$ is the y-coordinate of the point on the y-axis (the line $x=0$) where the lines $Ax+By=C$ and $x=0$ intersect. To find the y-intercept for the line $Ax+By=C$, we set $x=0$ and solve for y to obtain $y=\frac{C}{B}$ provided $B\neq 0$.
2. The x ***-intercept*** of the line $Ax+By=C$ is the x-coordinate of the point on the x-axis (the line $y=0$) where the lines $Ax+By=C$ and $y=0$ intersect. To find the x-intercept for the line $Ax+By=C$, we set $y=0$ and solve for x to obtain $x=\frac{C}{A}$ provided $A\neq 0$.

Example 4. Find the slope and the x-and y-intercepts of the line $2x-3y=6$ and sketch its graph. Find at least three points on the line and place the coordinates of these points on your graph.

Solution. We first find the x-and y-intercepts. We set $x=0$ in the equation $2x-3y=6$ and solve for y to obtain

$-3y=6$ Thus $y=-2$ is the y-intercept and $(0,-2)$ is on the line.

We set $y=0$ in the equation $2x-3y=6$ and solve for x to obtain

$2x=6$ Thus $x=3$ is the x-intercept and $(3,0)$ is on the line.

We arbitrarily choose a value of y, say $y=1$, substitute it into the equation $2x-3y=6$, and solve for x to obtain another point $\left(\frac{9}{2},1\right)$ on the line.

We can find the slope of the line using one of two methods. Both methods are useful depending on the problem being solved and so we discuss each of them.

Method 1. We select any two points on the line, $(0,-2)$ and $(3,0)$ in this case, and calculate the slope using them. If we go from $(0,-2)$ to $(3,0)$, the x-coordinate *increases* by 3 and the y-coordinate *increases* by 2. Therefore the slope of the line is

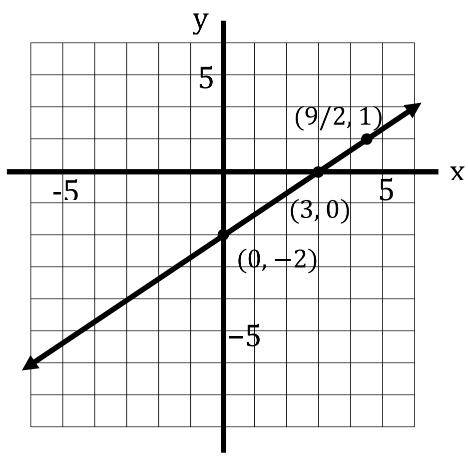
$$slope=\frac{change\ in\ y}{change\ in\ x}=\frac{2}{3}$$

Method 2. We solve $2x-3y=6$ for y to put the equation of the line in slope-intercept form.

$2x-3y=6$	Given equation
$-3y=-2x+6$	Subtract $2x$
$y=\frac{2}{3}x-2$	Divide by -3

Thus $y=\frac{2}{3}x-2$ is the equation of the line in slope-intercept form. The slope of the line is therefore $\frac{2}{3}$, confirming what we found using Method 1.

A sketch of the graph of the line with the three points included is shown below.

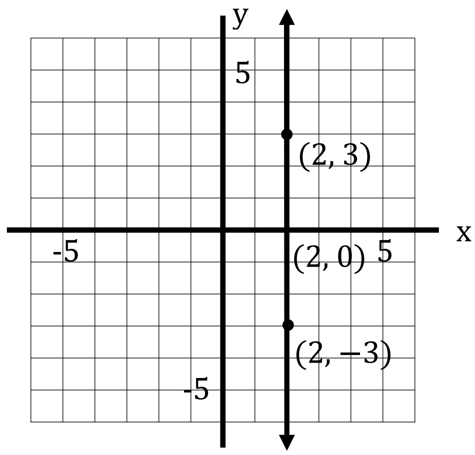


Example 5. Find the slope and the x- and y-intercepts of the line $3x=6$, if possible, and sketch its graph. Find at least three points on the line and place the coordinates of these points on the graph.

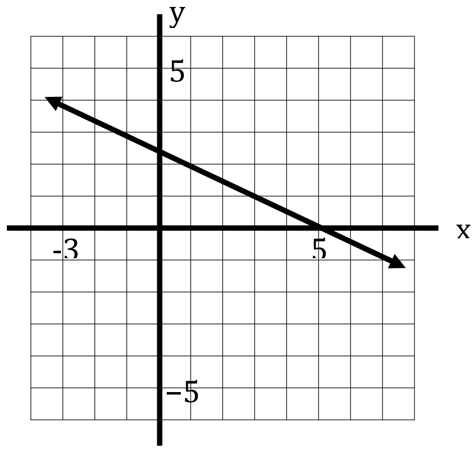
Solution. We see that $3x=6$ has no term involving y and is therefore a vertical line. If we think of this line as a linear equation in two variables with the coefficient of y equal to 0, we obtain

$$3x+0y=6$$

We see that any solution of this equation has x-coordinate equal 2 and y-coordinate that can be any number. Thus points $(2,-3)$, $(2,0)$ and $(2,3)$ all satisfy this equation and are points on the vertical line $x=2$ that is sketched below. The line has no slope, no y-intercept, and an x-intercept 2.



Example 6. Find the slope-intercept form of the equation of the line whose graph is shown.



Solution. We choose two points $(3,1)$ and $(5,0)$ on the graph that occur at the intersection of grid lines and so have integer coordinates. If we go from $(3,1)$ to $(5,0)$, the x-coordinate *increases* by 2 and the y-coordinate *decreases* by 1. Therefore the slope of the line is

$$slope = \frac{change\ in\ y}{change\ in\ x} = \frac{-1}{2} = -\frac{1}{2}$$

The equation of the line has the form

$$y = -\frac{1}{2}x + b$$

The variable b is the y-intercept.

Both the points $(3,1)$ and $(5,0)$ are on the line and satisfy this equation. If we use $(5,0)$, then we obtain

$0 = -\frac{1}{2}(5) + b$ We solve for b to obtain $b = \frac{5}{2}$.

The equation of the line in slope-intercept form is therefore $y = -\frac{1}{2}x + \frac{5}{2}$. Note that this is the line we discussed in Example 1.

Exercise Set 7.2

Find the x- and y-intercepts of the line with the given equation and sketch its graph. Calculate the slope of the line using the two points where the line intersects the axes. Check your answer by placing the equation of the line in slope-intercept form.

1. $2x + 3y = 6$

2. $3x - 4y = 12$

3. $-3x + 2y = 18$

4. $x + 2y = 8$

5. $2x + y = -4$

6. $x - 3y = -9$

Find the equation of the line in slope-intercept form that satisfies the given conditions.

7. Through $(-4, 4)$, with slope -1

8. Through $(1, 3)$ and $(5, 1)$

9. Through $(-2, 3)$ and $(3, -5)$

10. Through $(-1, 2)$ and $(6, 4)$

11. Through $(-1, -2)$, with slope $\frac{2}{5}$

12. Through $(4, -5)$, with slope $-\frac{3}{4}$

13. Slope 2, with x-intercept: -3

14. Slope $\frac{7}{3}$, with y-intercept: 1

15. x-intercept -1 and y-intercept 4

16. x-intercept 3 and y-intercept 5

Sketch the graph of the line. Find at least three points on the line and place the coordinates of these points on your graph.

17. $y = -x + 3$

18. $y = 2x - 5$

19. $y = -\frac{1}{3}x$

20. $y = \frac{5}{2}x + 1$

21. $y=\frac{2}{3}x-4$

22. $y=-\frac{3}{5}x-2$

Find the slope and the x- and y-intercepts of the line and sketch its graph. Find at least three points on the line and place the coordinates of these points on your graph.

23. $x+y=4$

24. $3x-2y=6$

25. $2x-5y=0$

26. $y=-3$

27. $5x+3y+15=0$

28. $5x+8=0$

29. $-2x+3y+9=0$

30. $4x+5y-10=0$

31. $3y-7=0$

32. $3y-7x=0$

33. $\frac{1}{3}x+\frac{1}{2}y=1$

34. $\frac{3}{4}x-\frac{1}{2}y=6$

35. $\frac{2}{5}x-y=1$

36. $2x+\frac{1}{3}y+4=0$

---

37. A line has equation $y=mx+1$, where m is a real number. What is the slope of the line if it passes through the point $(3,-1)$?

38. A line has equation $y=mx-5$, where m is a real number. What is the slope of the line if its x-intercept is 2?

39. A line has equation $ax+3y=5$, where a is a real number. What is the x-intercept of the line if it passes through the point $(2,-1)$?

40. A line has equation $2x+by=3$, where b is a real number. What is the slope of this line if it passes through the point $(-1,2)$?

41. A line has equation $y=-2x+b$, where b is a real number. What is the y-intercept of the line if its x-intercept is 3?

42. A line has equation $-2x+4y=c$, where c is a real number. What is the slope of the line if it passes through the point $(-1,-3)$?

43. A line with slope 2 passes through the point $(-3,-1)$. What is its y-intercept?

44. A line with slope -2 passes through the point $(2,3)$. What is its x-intercept?

45. A line with y-intercept -3 passes through the point $(-3,4)$. What is its slope?

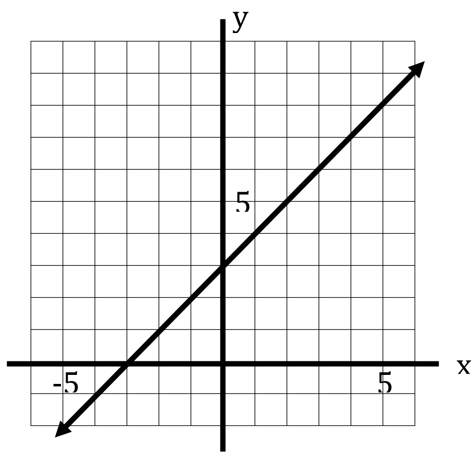
46. A line with x-intercept 4 passes through the point $(3,5)$. What is its slope?

47. A line with x-intercept -2 passes through the point $(3,3)$. What is its y-intercept?

48. A line with y-intercept -5 passes through the point $(-2,2)$. What is its x-intercept?

Find the slope-intercept form of the equation of the line whose graph is shown.

49.



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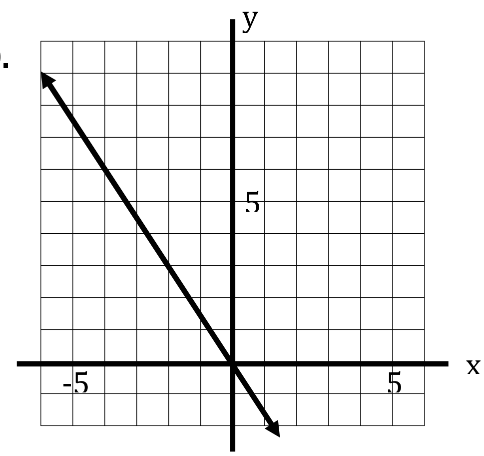

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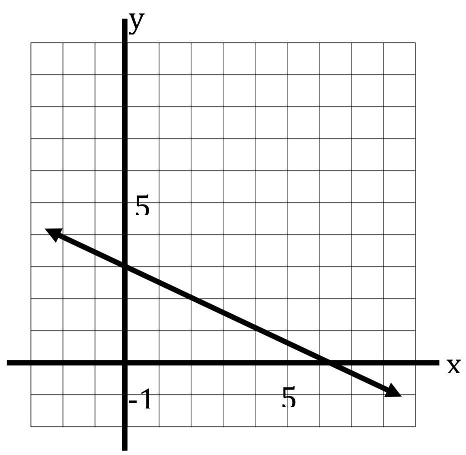

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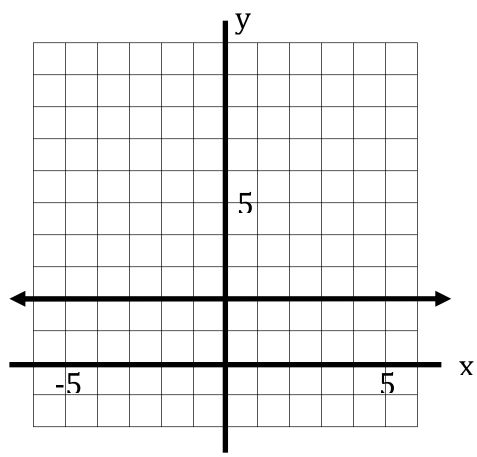

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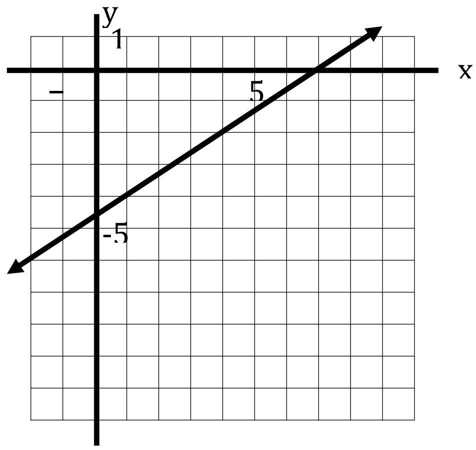

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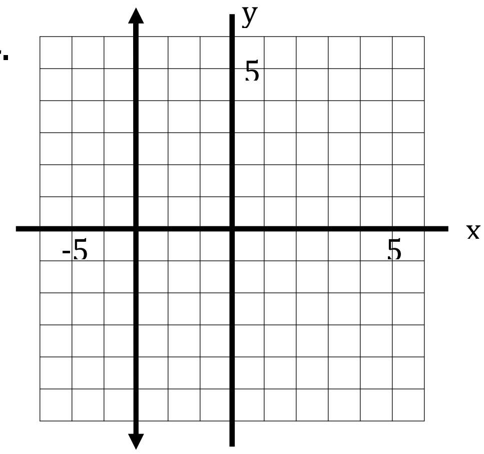

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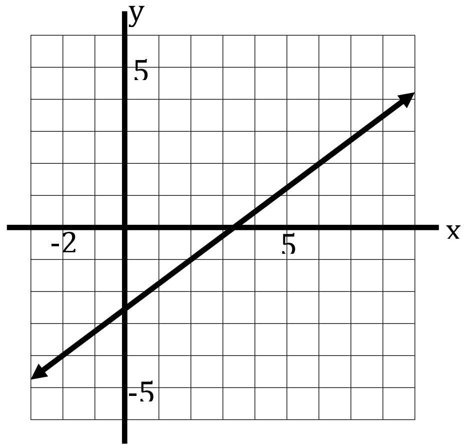

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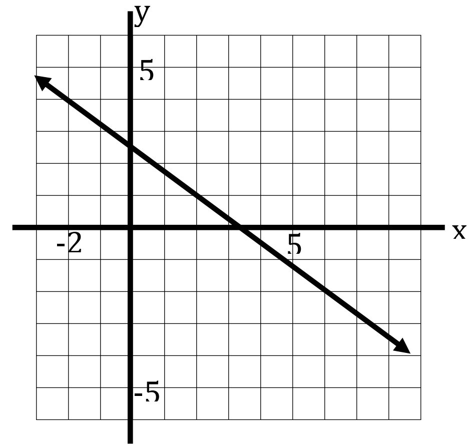


7.3. Parallel and Perpendicular Lines

KYOTE Standards: *CA 16*

We discuss parallel and perpendicular lines in this section.

Definition 1. Two nonvertical lines are *parallel* if and only if they have the same slope. Any two vertical lines are parallel.

Definition 2. Two lines with slopes m_1 and m_2 are *perpendicular* if and only if $m_1 m_2 = -1$; that is, their slopes are negative reciprocals:

$$m_2 = -\frac{1}{m_1}$$

Also, a horizontal line (slope 0) is perpendicular to a vertical line (no slope).

Example 1. Find an equation of the line through $(3,-4)$ that is parallel to the line $2x-5y=7$.

Solution. The line of interest passes through $(3,-4)$ and has the same slope as the line $2x-5y=7$ since these two lines are parallel. We put $2x-5y=7$ in slope-intercept form to find its slope.

$2x-5y=7$	Given equation
$-5y=-2x+7$	Subtract $2x$
$y=\frac{2}{5}x-\frac{7}{5}$	Divide by -5

The line of interest therefore has slope $\frac{2}{5}$ and can be written in slope-intercept form as

$y=\frac{2}{5}x+b$ — The variable b is the y-intercept.

The point $(3,-4)$ is on the line and thus satisfies this equation $y=\frac{2}{5}x+b$. We can use this to solve for b.

$-4=\frac{2}{5}(3)+b$ — We solve for b to obtain $b=-\frac{26}{5}$.

The line of interest thus has equation $y=\frac{2}{5}x-\frac{26}{5}$.

Example 2. Find an equation of the line through $(6,2)$ that is perpendicular to the line $y=\frac{3}{7}x+1$.

Solution. The line of interest passes through the point $(6,2)$ and is perpendicular to the line $y=\frac{3}{7}x+1$. The line $y=\frac{3}{7}x+1$ is in slope-intercept form and hence has slope $\frac{3}{7}$. The line of interest therefore has slope $-\frac{7}{3}$, the negative reciprocal of $\frac{3}{7}$. The line of interest can thus be written in slope-intercept form as

$$y=-\frac{7}{3}x+b$$ The variable b is the y-intercept.

The point $(6,2)$ is on this line and thus satisfies this equation $y=-\frac{7}{3}x+b$. We can use this to solve for b.

$$2=-\frac{7}{3}(6)+b$$ We solve for b to obtain $b=16$.

The line of interest thus has equation $y=-\frac{7}{3}x+16$.

Example 3. Find an equation of the line through $(3,-2)$ that is perpendicular to the line with x-intercept 5 and that passes through $(0,-4)$.

Solution. The line of interest passes through the point $(3,-2)$and has a slope equal to the negative reciprocal of the slope of the line passing through $(5,0)$ and $(0,-4)$. If we go from $(0,-4)$ to $(5,0)$, we see that the x-coordinate *increases* by 5 and the y-coordinate *increases* by 4, and thus this line has slope

$$slope=\frac{change\ in\ y}{change\ in\ x}=\frac{4}{5}$$

The line of interest has slope $-\frac{5}{4}$, the negative reciprocal of $\frac{4}{5}$. The line of interest can thus be written in slope-intercept form as

$y = -\frac{5}{4}x + b$ The variable b is the y-intercept.

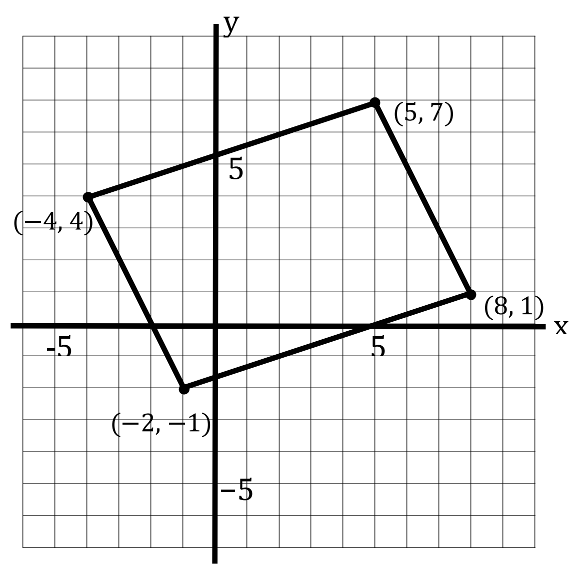
The point $(3,-2)$ is on this line and thus satisfies this equation $y = -\frac{5}{4}x + b$. We can use this to solve for b.

$-2 = -\frac{5}{4}(3) + b$ We solve for b to obtain $b = \frac{7}{4}$.

The line of interest thus has equation $y = -\frac{5}{4}x + \frac{7}{4}$.

Example 4. A quadrilateral has vertices $(-1,-2)$, $(8,1)$, $(5,7)$ and $(-4,4)$. Plot these points and sketch the quadrilateral. Use slopes to determine whether the quadrilateral is a parallelogram and whether the quadrilateral is a rectangle. Justify your answers.

Solution. We plot these points and sketch the quadrilateral as shown.



The line segment from $(-1,-2)$ to $(8,1)$ and the line segment from $(-4,4)$ to $(5,7)$ are parallel since both have slope $\frac{1}{3}$. The line segment from $(-1,-2)$ to $(-4,4)$ and the line segment from $(8,1)$ to $(5,7)$ are also parallel since both have slope -2. Since opposite sides of the quadrilateral are parallel, we conclude that the quadrilateral is a parallelogram.

A parallelogram is a rectangle if the angles formed by its sides are all right angles. The line segment form $(-1,-2)$ to $(-4,4)$ is not perpendicular to the line segment from $(-1,-2)$ to $(8,1)$ because the product of their slopes, $(-2)\cdot\frac{1}{3}=-\frac{2}{3}$, is *not* equal to -1. Hence the angle formed by these lines segments is *not* a right angle. Therefore the quadrilateral is *not* a rectangle.

Exercise Set 7.3

In exercises 1-8, determine whether the given lines are parallel, perpendicular, or neither. Justify your answer.

1. $x-4y=6$ *and* $-2x+8y=12$

2. $x+3y=9$ *and* $-2x-3y=18$

3. $4x+2y=10$ *and* $x-2y=4$

4. $3x-4y=12$ *and* $4x-3y=24$

5. $2x-3y=6$ *and* $y=\frac{3}{2}x+5$

6. $7x-5y=35$ *and* $y=-\frac{5}{7}x$

7. $3x+5y=15$ *and* $y=-\frac{3}{5}x+7$

8. $8x-3y=0$ *and* $y=-\frac{3}{8}x-12$

9. Find an equation of the line through $(-1,4)$ that is parallel to the line $y=3x-2$.

10. Find an equation of the line through $(-2,3)$ that is parallel to the line $y=7$.

11. Find an equation of the line through $(-2,3)$ that is parallel to the line $x=-5$.

12. Find an equation of the line through $(2,1)$ that is perpendicular to the line $y=\frac{2}{3}x+1$.

13. Find an equation of the line with x-intercept -5 that is parallel to the line $2x+3y=6$.

14. Find an equation of the line with y-intercept 2 that is perpendicular to the line $3x-4y=0$.

15. Find an equation of the line through $(1,2)$ that is parallel to the line $y=-\frac{3}{5}x-4$.

16. Find an equation of the line through $(-4,5)$ that is perpendicular to the y-axis.

17. Find an equation of the line through $(-4,5)$ that is perpendicular to the x-axis.

18. Find an equation of the line through $(4,3)$ that is parallel to $x+3y+1=0$.

19. Find an equation of the line through $(-1,-3)$ that is perpendicular to $-5x+3y-15=0$.

20. Find an equation of the line through $(0,0)$ that is parallel to the line through $(4,-1)$ and $(-6,5)$.

21. Find an equation of the line through $(5,4)$ that is perpendicular to the line through $(-1,2)$ and $(3,7)$.

22. Find an equation of the line through $(-3,-4)$ that is perpendicular to the line through $(2,5)$ and $(2,0)$.

23. Find an equation of the line through $(-3,-4)$ that is parallel to the line through $(2,5)$ and $(2,0)$.

24. Find an equation of the line with x-intercept 2 that is parallel to the line through $(-1,-2)$ and $(3,5)$.

25. Find an equation of the line with y-intercept 3 that is perpendicular to the line through $(2,-3)$ and $(-2,2)$.

26. Find an equation of the line with x-intercept 2 perpendicular to the line through $(-1,4)$ and $(3,4)$.

27. A quadrilateral has vertices $(2,2)$, $(8,2)$, $(10,5)$ and $(4,5)$. Plot these points and sketch the quadrilateral. Use slopes to determine whether the quadrilateral is a parallelogram. Justify your answer.

28. A quadrilateral has vertices $(1,1)$, $(7,4)$, $(5,10)$ and $(-1,7)$. Plot these points and sketch the quadrilateral. Use slopes to determine whether the quadrilateral is a parallelogram. Justify your answer.

29. A quadrilateral has vertices $(-1,-2)$, $(6,0)$, $(8,7)$ and $(1,4)$. Plot these points and sketch the quadrilateral. Use slopes to determine whether the quadrilateral is a parallelogram. Justify your answer.

30. A triangle has vertices $(0,0)$, $(4,1)$ and $(-2,8)$. Plot these points and sketch the triangle. Use slopes to determine whether the triangle is a right triangle. Justify your answer.

31. A triangle has vertices $(-3,-1)$, $(3,3)$ and $(-9,8)$. Plot these points and sketch the triangle. Use slopes to determine whether the triangle is a right triangle. Justify your answer.

32. A triangle has vertices $(2,1)$, $(4,-1)$ and $(4,5)$. Plot these points and sketch the triangle. Use slopes to determine whether the triangle is a right triangle. Justify your answer.

33. A quadrilateral has vertices $(1,1)$, $(3,3)$, $(2,4)$ and $(0,2)$. Plot these points and sketch the quadrilateral. Use slopes to determine whether the quadrilateral is a rectangle. Justify your answer.

34. Plot the points $(1,0)$, $(3,4)$ and $(6,10)$. Use slopes to determine whether these points lie on the same line. Justify your answer.

35. Plot the points $(1,4)$, $(3,2)$ and $(6,-2)$. Use slopes to determine whether these points lie on the same line. Justify your answer.

Chapter 8. Quadratic Equations and Functions

8.1. Solve Quadratic Equations

KYOTE Standards: *CR 20; CA 11*

In this section, we discuss solving quadratic equations by factoring, by using the square root property and by using the quadratic formula. We begin by defining a quadratic equation and what it means for a number to be a solution, or root, of the quadratic equation.

Definition 1. A *quadratic equation in one variable* is an equation that can be written in the form $ax^2 + bx + c = 0$, where x is the variable, a, b and c are real numbers and $a \neq 0$. A *solution*, or *root*, of this equation is a value of x that makes $ax^2 + bx + c = 0$ a true statement. Such a number is said to *satisfy* the equation.

Factoring

The key to solving quadratic equations by factoring depends upon the *zero-product property* of real numbers.

Zero-Product Property

If a and b are real numbers such that $ab = 0$, then either $a = 0$ or $b = 0$.

Example 1. Solve $2x^2 + x - 15 = 0$ by factoring.

Solution. If we can factor the quadratic polynomial $2x^2 + x - 15$, then it is fairly easy to find its roots using the zero-product property. We obtain

$2x^2 + x - 15 = 0$	Given equation
$(2x-5)(x+3) = 0$	Factor $2x^2 + x - 15$
$2x - 5 = 0 \quad or \quad x + 3 = 0$	Zero-product property
$x = \frac{5}{2} \quad or \quad x = -3$	Solve both linear equations

The solutions (roots) of the quadratic equation are $x = \frac{5}{2}$ and $x = -3$. We can check our answers by substituting back into the original equation. We obtain

$$2\left(\frac{5}{2}\right)^2 + \frac{5}{2} - 15 = \frac{25}{2} + \frac{5}{2} - 15 = \frac{30}{2} - 15 = 15 - 15 = 0$$

$$2(-3)^2 + (-3) - 15 = 18 - 3 - 15 = 0$$

Square-Root Property

Quadratic equations of the form $(x-p)^2 = q$, where p and q are real numbers and $q > 0$ can be solved easily using the square root property. In the next section, we show that *any* quadratic equation can be put in this form and this is the key to deriving the familiar quadratic formula for solving *any* quadratic equation.

Square Root Property

Suppose x satisfies the quadratic equation $(x-p)^2 = q$, where p and q are real numbers and $q > 0$. Then $x - p = \sqrt{q}$ or $x - p = -\sqrt{q}$.

Example 2. Solve $2(x-5)^2 - 14 = 0$ by using the square root property.

Solution. We put $2(x-5)^2 - 14 = 0$ in the form $(x-p)^2 = q$ and use the square root property to find its two solutions.

$2(x-5)^2 - 14 = 0$	Given equation
$2(x-5)^2 = 14$	Add 14
$(x-5)^2 = 7$	Divide by 2
$x - 5 = \sqrt{7}$ *or* $x - 5 = -\sqrt{7}$	Square root property
$x = 5 + \sqrt{7}$ *or* $x = 5 - \sqrt{7}$	Solve both linear equations

The solutions of the quadratic equation are $x = 5 + \sqrt{7}$ and $x = 5 - \sqrt{7}$.

Quadratic Formula

Any quadratic equation $ax^2 + bx + c = 0$ can be solved in terms of its coefficients a, b and c using the quadratic formula.

Quadratic Formula

The roots of the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are real numbers with $a \neq 0$, can be found using the *quadratic formula* $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $D = b^2 - 4ac$ of the quadratic equation $ax^2 + bx + c = 0$ can be used to determine how many real number solutions this equation has.

The Discriminant

The *discriminant* of the quadratic equation $ax^2+bx+c=0$ is $D=b^2-4ac$. The discriminant can be used to determine how many real number solutions the quadratic equation has.

1. If $D>0$, then the equation has two distinct real number solutions.
2. If $D=0$, then the equation has exactly one real number solution.
3. If $D<0$, then the equation has no real number solution.

Example 3. Solve $x^2-6x+7=0$ using the quadratic formula.

Solution. We identify the coefficients $a=1$, $b=-6$ and $c=7$ to be used in the quadratic formula. We then substitute these numbers into the quadratic formula to obtain

$x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(1)(7)}}{2(1)}$	Quadratic formula: $a=1$, $b=-6$, $c=7$
$x=\dfrac{6\pm\sqrt{8}}{2}$	Simplify
$x=\dfrac{6\pm 2\sqrt{2}}{2}$	Simplify $\sqrt{8}=\sqrt{4\cdot 2}=2\sqrt{2}$
$x=3\pm\sqrt{2}$	Divide by 2

The solutions of the quadratic equation are $x=3+\sqrt{2}$ and $x=3-\sqrt{2}$.

Example 4. Use the discriminant of the given quadratic equation to determine how many real number solutions it has. If it has real solutions, use the quadratic formula to find them.

(a) $2x^2-x-3=0$ **(b)** $4x^2-12x+9=0$ **(c)** $x^2+2x+5=0$

Solution. **(a)** We identify the coefficients $a=2$, $b=-1$ and $c=-3$ to be used in the formula for the discriminant and evaluate it to obtain

$D=b^2-4ac$	Discriminant formula
$D=(-1)^2-4(2)(-3)$	Substitute: $a=2$, $b=-1$, $c=-3$
$D=25$	Evaluate D

Since $D>0$, the quadratic equation $2x^2-x-3=0$ has two solutions and we can find these solutions using the quadratic formula. We obtain

$x=\dfrac{-(-1)\pm\sqrt{25}}{2(2)}$	Quadratic formula: $a=2$, $b=-1$, $c=-3$, $D=25$
$x=\dfrac{1\pm 5}{4}$	Simplify

$x = \frac{1+5}{4}$ *or* $x = \frac{1-5}{4}$ Use $+$ to get one solution and $-$ to get the other

The two solutions are $x = \frac{3}{2}$ and $x = -1$.

(b) We identify the coefficients $a = 4$, $b = -12$ and $c = 9$ to be used in the formula for the discriminant and evaluate it to obtain

$D = b^2 - 4ac$ Discriminant formula

$D = (-12)^2 - 4(4)(9)$ Substitute: $a = 4$, $b = -12$, $c = 9$

$D = 0$ Evaluate D

Since $D = 0$, the quadratic equation $4x^2 - 12x + 9 = 0$ has only one solution and we can find it using the quadratic formula. We obtain

$x = \frac{-(-12) \pm \sqrt{0}}{2(4)}$ Quadratic formula: $a = 4$, $b = -12$, $c = 9$, $D = 0$

$x = \frac{3}{2}$ Simplify

The solution is $x = \frac{3}{2}$.

(c) We identify the coefficients $a = 1$, $b = 2$ and $c = 5$ to be used in the formula for the discriminant and evaluate it to obtain

$D = b^2 - 4ac$ Discriminant formula

$D = 2^2 - 4(1)(5)$ Substitute: $a = 1$, $b = 2$, $c = 5$

$D = -16$ Evaluate D

Since $D < 0$, the quadratic equation $x^2 + 2x + 5 = 0$ has no real number solutions. If the discriminant D in the quadratic formula is negative, we cannot take its square root and get a real number. This is the reason the equation has no real number solutions.

The method we use to solve a quadratic equation is often not specified. In this case we can select whatever method works and is most convenient.

Example 5. Solve the quadratic equation $x^2 + 3x = 10$.

Solution. We first place the quadratic equation in form

$$x^2 + 3x - 10 = 0$$

It appears that this equation will be easy to factor and try that approach. We obtain

$x^2+3x-10=0$	Given equation
$(x+5)(x-2)=0$	Factor $x^2+3x-10$
$x+5=0 \quad or \quad x-2=0$	Zero-product property
$x=-5 \quad or \quad x=2$	Solve both linear equations

The two solutions are $x=-5$ and $x=2$.

We could use the quadratic formula and get the same solutions if we choose to do so. We first identify the coefficients $a=1$, $b=3$ and $c=-10$ to be used. We then substitute these numbers into the quadratic formula to obtain

$x=\dfrac{-3\pm\sqrt{3^2-4(1)(-10)}}{2(1)}$	Quadratic formula: $a=1$, $b=3$, $c=-10$
$x=\dfrac{-3\pm\sqrt{49}}{2}=\dfrac{-3\pm 7}{2}$	Simplify
$x=\dfrac{-3+7}{2} \quad or \quad x=\dfrac{-3-7}{2}$	Use $+$ to get one solution and $-$ to get the other

The two solutions are $x=2$ and $x=-5$, confirming what we obtained by factoring.

Example 6. Solve the quadratic equation $(x+2)^2=9$.

Solution. We have two options in this case. We could square $x+2$ and put the equation in the form in which we could solve it by factoring or by the quadratic formula.

The simpler option is to take advantage of the structure of the equation and use the square root property. We obtain

$(x+2)^2=9$	Given equation
$x+2=3 \quad or \quad x+2=-3$	Square root property; note $\sqrt{9}=3$
$x=1 \quad or \quad x=-5$	Solve both linear equations

The two solutions are $x=1$ and $x=-5$.

Exercise Set 8.1

Solve the quadratic equation by factoring. Solve the same equation by using the quadratic formula to check your answers.

1. $x^2 - 5x + 6 = 0$

2. $x^2 - 2x = 3$

3. $x^2 + 7x + 12 = 0$

4. $x^2 + 4x + 4 = 0$

5. $x^2 + 5x - 14 = 0$

6. $x^2 = 4x + 12$

7. $2x^2 + 5x - 3 = 0$

8. $3x^2 - x - 2 = 0$

Use the discriminant of the given quadratic equation to determine how many real number solutions it has. If it has real solutions, use the quadratic formula to find them.

9. $x^2 + 2x - 5 = 0$

10. $x^2 - 3x = 1$

11. $x^2 + 8 = 0$

12. $x^2 - 6x + 9 = 0$

13. $x^2 + 6x + 1 = 0$

14. $x^2 - 6x + 4 = 0$

15. $2x^2 + 4x + 2 = 0$

16. $2x^2 = 3x + 5$

17. $3x^2 - 2x + 1 = 0$

18. $3x^2 = 6x - 9$

Solve the given quadratic equation by finding all its real number solutions.

19. $x^2 - 9 = 0$

20. $x^2 - 5x = 0$

21. $x^2 + x - 6 = 0$

22. $x^2 - 8x + 15 = 0$

23. $x^2 + 11x + 28 = 0$

24. $(x-1)^2 - 4 = 0$

25. $x^2 + 3x + 1 = 0$

26. $2x^2 + x - 3 = 0$

27. $2x^2 - 16 = 0$

28. $4x^2 - 4x - 15 = 0$

29. $\left(x + \frac{1}{2}\right)^2 = 1$

30. $x^2 - 2x + 2 = 0$

31. $x^2 - 4x + 4 = 0$

32. $x^2 - 4x + 2 = 0$

33. $(x-5)^2 = 2$

34. $x^2 + 8 = 0$

35. $x^2 = 6x - 9$

36. $3x^2 - 13x - 10 = 0$

37. $\left(x - \frac{3}{2}\right)^2 - \frac{1}{4} = 0$

38. $x^2 + x = 2$

39. $3x^2 - 5x - 1 = 0$

40. $2x^2 - x - \frac{1}{2} = 0$

41. $4x^2 - x - 5 = 0$

42. $4(x+1)^2 = 3$

43. $(x+2)^2 + 4 = 0$

44. $x^2 - 2x - 35 = 0$

8.2. Completing the Square

KYOTE Standards: *CA 11, CA 17*

The technique of ***completing the square*** can be used to transform *any* quadratic equation to an equivalent equation in the form $(x-p)^2 = q$, where p and q are real numbers. We saw in the last section how an equation in this form can be solved using the square root property. This technique is used to derive the quadratic formula.

This technique is also used to write a quadratic function in a form that enables us to sketch its graph without difficulty. We shall see how this is done in the next section.

We begin by highlighting the key idea that is used in completing the square.

General Approach to Completing the Square

To complete the square for the quadratic expression $x^2 + bx$, where b a real number, we add and subtract $\left(\frac{b}{2}\right)^2$ to obtain

$$\begin{aligned} x^2 + bx &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \end{aligned}$$

We can use this approach to complete the square for any quadratic expression and solve any quadratic equation as is shown in the following examples.

Example 1. Complete the square for the given quadratic expression.

(a) $x^2 - 3x$ **(b)** $2x^2 + 4x$ **(c)** $-x^2 + x$

Solution. **(a)** We follow the general approach highlighted above.

$x^2 - 3x$	Given expression
$= \left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4}$	Add and subtract $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$
$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$	Write $x^2 - 3x + \frac{9}{4} = \left(x - \frac{3}{2}\right)^2$

Therefore $x^2 - 3x = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$ after completing the square.

(b) Since the coefficient of x^2 in the expression $2x^2+4x$ is not 1, we cannot complete the square we did in part (a). Instead, we first factor out 2 and then follow the general approach highlighted above.

$2x^2+4x$	Given expression
$=2(x^2+2x)$	Factor out 2
$=2\left(x^2+2x+1-1\right)$	Add and subtract $\left(\frac{2}{2}\right)^2=1$
$=2\left((x+1)^2-1\right)$	Write $x^2+2x+1=(x+1)^2$
$=2(x+1)^2-2$	Multiply by 2

Therefore $2x^2+4x=2(x+1)^2-2$ after completing the square.

(c) Since the coefficient of x^2 in the expression $-x^2+x$ is not 1, we first factor out -1 and then follow the general approach highlighted above.

$-x^2+x$	General expression
$=-(x^2-x)$	Factor out -1
$=-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)$	Add and subtract $\left(-\frac{1}{2}\right)^2=\frac{1}{4}$
$=-\left(\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right)$	Write $x^2-x+\frac{1}{4}=\left(x-\frac{1}{2}\right)^2$
$=-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}$	Multiply by -1

Therefore $-x^2+x=-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}$ after completing the square.

The approach we take to solve a quadratic equation by completing the square is essentially the same as the approach we take to complete the square for a quadratic expression. But there are some minor differences that are highlighted in Example 2 that arise because we are dealing with equations rather than expressions.

Example 2. Solve the given quadratic equation by completing the square.

(a) $x^2+6x-7=0$ **(b)** $x^2-2x-1=0$ **(c)** $4x^2-12x+5=0$

***Solution*. (a)** We solve $x^2+6x-7=0$ by completing the square as follows:

$x^2+6x-7=0$	Given equation
$x^2+6x=7$	Add 7

$x^2+6x+9=7+9$ Add $\left(\frac{6}{2}\right)^2=9$

Note that we *added* 9 to both sides of the equation instead of adding and subtracting 9 from x^2+6x as we would do when completing the square for the quadratic expression x^2+6x.

$(x+3)^2=16$ Write $x^2+6x+9=(x+3)^2$

$x+3=4 \quad or \quad x+3=-4$ Square root property

$x=1 \quad or \quad x=-7$ Solve both linear equations

The solutions of $x^2+6x-7=0$ are $x=1$ and $x=-7$.

(b) We solve $x^2-2x-1=0$ by completing the square as follows:

$x^2-2x-1=0$ Given equation

$x^2-2x=1$ Add 1

$x^2-2x+1=1+1$ Add $\left(\frac{-2}{2}\right)^2=1$

Note that we *added* 1 to both sides of the equation instead of adding and subtracting 1 from x^2-2x as we would do when completing the square for the quadratic expression x^2-2x.

$(x-1)^2=2$ Write $x^2-2x+1=(x-1)^2$

$x-1=\sqrt{2} \quad or \quad x-1=-\sqrt{2}$ Square root property

$x=1+\sqrt{2} \quad or \quad x=1-\sqrt{2}$ Solve both linear equations

The solutions of $x^2-2x-1=0$ are $x=1+\sqrt{2}$ and $x=1-\sqrt{2}$.

(c) We solve $4x^2-12x+5=0$ by completing the square as follows:

$4x^2-12x+5=0$ Given equation

$4x^2-12x=-5$ Subtract 5

$x^2-3x=-\frac{5}{4}$ Divide by 4

Note that we divide both sides of the equation by 4 instead of factoring out 4 as we would do when completing the square for the quadratic expression $4x^2-12x$.

$x^2-3x+\frac{9}{4}=-\frac{5}{4}+\frac{9}{4}$ Add $\left(\frac{-3}{2}\right)^2=\frac{9}{4}$

$\left(x-\frac{3}{2}\right)^2=1$ Write $x^2-3x+\frac{9}{4}=\left(x-\frac{3}{2}\right)^2$

$x-\frac{3}{2}=1 \quad or \quad x-\frac{3}{2}=-1$ Square root property

$x=\frac{5}{2} \quad or \quad x=\frac{1}{2}$ Solve both linear equations

The solutions of $4x^2-12x+5=0$ are $x=\frac{5}{2}$ and $x=\frac{1}{2}$.

Exercise Set 8.2

Complete the square for the given quadratic expression

1. x^2+2x

2. x^2-4x

3. x^2+5x

4. x^2-7x

5. x^2-x

6. $2x^2+12x$

7. $3x^2-12x$

8. $3x^2+9x$

9. $4x^2+24x$

10. $2x^2+x$

11. $2x^2-5x$

12. $3x^2-2x$

13. $-x^2+4x$

14. $-2x^2+3x$

Solve the given quadratic equation by completing the square

15. $x^2+2x=0$

16. $x^2-x=0$

17. $x^2-2x-3=0$

18. $x^2-4x+3=0$

19. $x^2+6x+4=0$

20. $2x^2-4x-1=0$

21. $4x^2-4x-3=0$

22. $x^2-4x-2=0$

23. $2x^2+8x+1=0$

24. $3x^2-6x-1=0$

25. $x^2+4x-1=0$

26. $4x^2-3x-4=0$

27. $x^2+2x+5=0$

28. $(x+3)(x-1)=1$

29. $4x^2-4x+1=0$

30. $(x-2)(x+4)=-7$

31. $x^2-2x+3=0$

32. $x^2+5x+5=0$

8.3. Graph Quadratic Functions

KYOTE Standards: *CA 17*

In the previous section, we saw how the technique of completing the square for a quadratic expression could be used to solve a quadratic equation. We use this technique in this section to sketch the graph a quadratic function.

A quadratic function associates with each real number x exactly one number y given by $y = ax^2 + bx + c$, where $a \neq 0$, b and c are real numbers. Each such pair of numbers (x, y) forms a graph in the plane called a *parabola*. This parabola is in general difficult to sketch. But it can be can be easily sketched if we can put the quadratic function in *vertex form* by completing the square.

Definition 1. The ***vertex form*** of a quadratic function is

$$y = m(x-h)^2 + k$$

where $m \neq 0$, h and k are real numbers.

Suppose a quadratic function is in vertex form $y = m(x-h)^2 + k$. Its graph is easy to construct. Its vertex is at the point (h, k) and it is symmetric around the line $x = h$. It opens upward if $m > 0$ and has a minimum value k at $x = h$. It opens downward if $m < 0$ and has a maximum value k at $x = h$. The examples show how this works.

Example 1. Consider the quadratic function $y = 2(x-1)^2 - 4$ in vertex form.

(a) Find its vertex and its minimum value.
(b) Find its x- and y-intercepts.
(c) Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.

Solution. **(a)** The vertex of $y = 2(x-1)^2 - 4$ is $(1, -4)$. Since $2(x-1)^2$ is 0 when $x = 1$ and positive when $x \neq 1$, we see that the graph of the function is a parabola that has a minimum value of -4 when $x = 1$. We also see that it is symmetric about the line $x = 1$ and opens upward. We plot some points to help visualize these properties although you are not required to do this in the exercises.

x	$y = 2(x-1)^2 - 4$
–1	4
0	–2
1	–4
2	–2
3	4

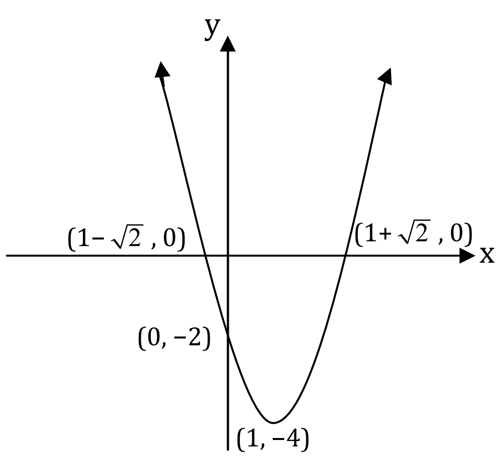
(b) We set $x=0$ in $y=2(x-1)^2-4$ to obtain $y=-2$. Thus the y-intercept is -2 and $(0,-2)$ is a point on the parabola, as is confirmed in the table above.

We set $y=0$ in $y=2(x-1)^2-4$ and solve the equation $2(x-1)^2-4=0$ for x to find the x-intercepts using the square root property.

$2(x-1)^2-4=0$	Given equation
$2(x-1)^2=4$	Add 4
$(x-1)^2=2$	Divide by 2
$x-1=\sqrt{2}$ *or* $x-1=-\sqrt{2}$	Square root property
$x=1+\sqrt{2}$ *or* $x=1-\sqrt{2}$	Solve both linear equations

The x-intercepts are $x=1+\sqrt{2}$ and $x=1-\sqrt{2}$, and the points $(1+\sqrt{2},0)$ and $\left(1-\sqrt{2},0\right)$ are points on the parabola.

(c) The sketch of the graph, with the coordinates of the points corresponding to the vertex and the intercepts on the graph, is shown below.



Example 2. Consider the quadratic function $y=-x^2+4x+5$.

(a) Write $y=-x^2+4x+5$ in vertex form by completing the square.

(b) Find its vertex and its maximum value.

(c) Find its x- and y-intercepts.

(d) Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.

Solution. **(a)** We write $y=-x^2+4x+5$ in vertex form by first completing the square for the quadratic expression $-x^2+4x$ as we did in the previous section. We obtain

$-x^2+4x$	Given expression
$=-(x^2-4x)$	Factor out -1
$=-\left(x^2-4x+4-4\right)$	Add and subtract $\left(\frac{-4}{2}\right)^2=4$
$=-\left((x-2)^2-4\right)$	Write $x^2-4x+4=(x-2)^2$
$=-(x-2)^2+4$	Multiply by -1

We replace $-x^2+4x$ by $-(x-2)^2+4$ in $y=-x^2+4x+5$ to obtain

$$y=(-x^2+4x)+5$$
$$y=\left(-(x-2)^2+4\right)+5$$
$$y=-(x-2)^2+9$$

The vertex form of $y=-x^2+4x+5$ is therefore $y=-(x-2)^2+9$.

(b) The vertex of $y=-(x-2)^2+9$ is $(2,9)$. Since $-(x-2)^2$ is 0 when $x=2$ and negative when $x\neq 2$, we see that the graph of the function has a maximum value of 9 when $x=2$. We also see it is a parabola symmetric about the line $x=2$ that opens downward. You can plot some points as in Example 1 to visualize these properties.

(c) We set $x=0$ in either one of the two forms of the function, $y=-x^2+4x+5$ or $y=-(x-2)^2+9$. We see in either case that the y-intercept is 5 and that $(0,5)$ is a point on the parabola.

We set $y=0$ in either one of the two forms of the function, $y=-x^2+4x+5$ or $y=-(x-2)^2+9$. We then solve either one of the quadratic equations, $-x^2+4x+5=0$ or $-(x-2)^2+9=0$, to find the x-intercepts.

The equation in vertex form is usually easier to solve since we can apply the square root property. But it is instructive in our example to solve both equations and confirm that the answers in each case are the same.

$-x^2+4x+5=0$	Given equation
$x^2-4x-5=0$	Multiply by -1
$(x+1)(x-5)=0$	Factor
$x+1=0 \quad or \quad x-5=0$	Zero-product property
$x=-1 \quad or \quad x=5$	Solve both linear equations

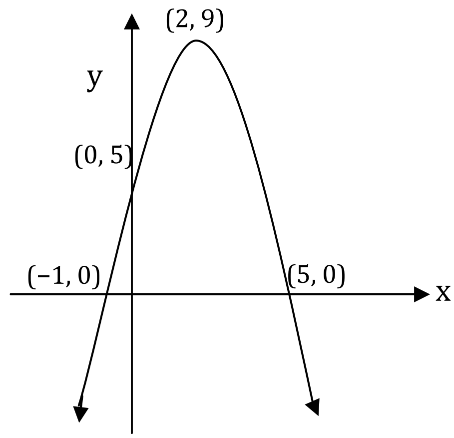
The solutions are $x = -1$ and $x = 5$.

We confirm this answer by solving $-(x-2)^2 + 9 = 0$.

$-(x-2)^2 + 9 = 0$	Given equation
$(x-2)^2 = 9$	Subtract 9; multiply by -1
$x - 2 = 3 \quad or \quad x - 2 = -3$	Square root property
$x = 5 \quad or \quad x = -1$	Solve both linear equations

The solutions are confirmed to be $x = -1$ and $x = 5$. Thus the x-intercepts are -1 and 5, and the points $(-1, 0)$ and $(5, 0)$ are points on the parabola.

(d) The sketch of the graph, with the coordinates of the points corresponding to the vertex and the intercepts on the graph, is shown below.



Exercise Set 8.3

A quadratic function in vertex form is given in exercises 1-12. In each case:

(a) *Find its vertex and its maximum or minimum value.*

(b) *Find its x- and y-intercepts.*

(c) *Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.*

1. $y = (x+2)^2 - 9$

2. $y = (x-5)^2 - 4$

3. $y = -(x+2)^2 + 1$

4. $y = 2(x-3)^2 - 8$

5. $y = 2(x+1)^2 + 4$

6. $y = 3(x-2)^2 - 6$

7. $y = -2(x+1)^2 + 5$

8. $y = -(x-4)^2 - 1$

9. $y=-\frac{1}{2}\left(x+\frac{3}{2}\right)^2+\frac{15}{8}$

10. $y=\left(x-\frac{5}{2}\right)^2-\frac{9}{4}$

11. $y=4(x+1)^2-8$

12. $y=\frac{1}{2}\left(x-\frac{3}{2}\right)^2-\frac{25}{8}$

A quadratic function is given in exercises 13-24. In each case:

(a) *Write its equation in vertex form by completing the square.*
(b) *Find its vertex and its maximum or minimum value.*
(c) *Find its* x*- and* y*-intercepts.*
(d) *Sketch its graph. Place the coordinates of the points corresponding to the vertex and the intercepts on the graph.*

13. $y=x^2+4x$

14. $y=x^2-2x-3$

15. $y=-x^2+10x$

16. $y=x^2+4x+3$

17. $y=3x^2-6x-1$

18. $y=x^2+x-1$

19. $y=-x^2+4x-1$

20. $y=-2x^2+8x-6$

21. $y=-x^2+4x-5$

22. $y=2x^2+6x$

23. $y=x^2+2x-2$

24. $y=2x^2+4x+3$

8.4. Applications of Quadratic Equations

KYOTE Standards: *CA 14*

Solving applied problems with quadratic equations involves the same five-step approach as solving applied problems with linear equations that we considered in Section 6.3. We state this five-step approach with "linear" replaced by "quadratic."

Five Steps to Use as Guidelines in Solving Applied Problems with Equations

1. **Define the Variable.** Read the problem carefully. The problem asks you to find some quantity or quantities. Choose one of these quantities as your variable and denote it by a letter, often the letter x. Write out a clear description of what the quantity x represents.
2. **Express All Other Unknown Quantities in Terms of the Variable.** Read the problem again. There are generally unknown quantities in the problem other than the one represented by the variable, say x. Express these unknown quantities in terms of x.
3. **Set up the Equation.** Set up a quadratic equation that gives a relationship between the variable and the unknown quantities identified in Step 2.
4. **Solve the Equation.** Solve the quadratic equation you obtain.
5. **Interpret Your Answer.** Write a sentence that answers the question posed in the problem. *Caution*. The variable name x or other unknown quantities expressed in terms of x should not appear in your interpretation.

Example 1. A rectangular room 3 feet longer than it is wide is to be carpeted with carpet costing \$2.50 per square foot. The total cost of the carpet is \$675. What are the length and width of the room?

Solution. We are asked to find the length and the width of the room. So we let

x =width of the room in feet — Step 1

The length of the room in terms of x can be written

$x+3$=length of the room in feet

Since the total cost of the carpet depends on the area of the room, we also write the area of the room in terms of x.

$x(x+3)$=area of the room in square feet — Step 2

We can find the total cost of the carpet by multiplying the area of the room in square feet by the cost of the carpet per square foot. The total cost is therefore

$$2.50\frac{dollars}{ft^2} \times x(x+3)\ ft^2 = 2.50x(x+3)\ dollars$$

The relationship between the area of the room and the cost of carpeting it can be written as an equation.

$$2.50\,x(x+3) = 675$$ Step 3

We solve this quadratic equation. We first simplify it by dividing both sides by 2.50 to obtain

$$x(x+3) = 270$$

> Note that we could also have obtained this equation by finding the area of the rectangle by dividing the total cost 675 by the cost 2.50 per square foot.
>
> $$\frac{675\,dollars}{2.50\,\frac{dollars}{ft^2}} = 270\ ft^2$$

We solve this equation using the quadratic formula after identifying the needed coefficients.

$x(x+3) = 270$	Given equation
$x^2 + 3x = 270$	Expand
$x^2 + 3x - 270 = 0$	Subtract 270

The coefficients of $x^2 + 3x - 270 = 0$ are $a = 1$, $b = 3$ and $c = -270$. We substitute these numbers into the quadratic formula to obtain

$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-270)}}{2(1)}$	Quadratic formula: $a = 1$, $b = 3$, $c = -270$
$x = \frac{-3 \pm \sqrt{1089}}{2}$	Simplify
$x = \frac{-3 \pm 33}{2}$	Simplify $\sqrt{1089} = 33$

The solutions are $x = 15$ and $x = -18$. Step 4

Note that we could also have solved the equation $x^2+3x-270=0$ by factoring to obtain $(x-15)(x+18)=0$, thus giving the same solutions $x=15$ and $x=-18$. But factoring in general becomes increasingly difficult as the coefficients of the quadratic function become larger.

The interpretation is particularly important in this problem. We reject the solution $x=-18$ because we cannot have a width of -18 feet! Thus the relevant solution in this problem is $x=15$, which is the width of the room in feet. The length of the room is $x+3=18$, which is also measured in feet.

We interpret the answer. Step 5

The length of the room is 18 feet and the width is 15 feet.

Exercise Set 8.4

KYOTE Standards: CA 14

1. The sum of the squares of two consecutive odd integers is 74. Find the two integers.

2. The sum of squares of two consecutive integers is 85. Find the two integers.

3. Two numbers have a sum of 21 and a product of 104. Find the numbers.

4. The length of a rectangular garden is twice its width and its area is 578 square feet. What are the length and width of the garden?

5. One leg of a right triangle is three times longer than its other leg. What are the lengths of the two legs if the hypotenuse is $7\sqrt{10}$ inches long?

6. The hypotenuse of a right triangle is twice as long as one of its legs and the other leg has length $9\sqrt{3}$ centimeters. What is the length of the hypotenuse?

7. One leg of a right triangle is 3 times longer than its other leg. What are the lengths of the two legs and the hypotenuse if the area of the triangle is 96 square inches?

8. A rectangle is 10 meters longer than it is wide and its area is 875 square meters. What are the length and width of the rectangle?

9. A rectangle of length 8 inches and width 5 inches is cut from a square piece of cardboard. If the area of the remaining cardboard is 321 square inches, what is the length of the square piece of cardboard?

10. The length of a rectangle is 2 feet longer than it is wide and its area is 224 square feet. What is its perimeter?

11. A rectangle is 24 feet long. The length of the diagonal between opposite corners of the rectangle is 12 feet more than its width. What is its width?

12. A rectangle has a perimeter of 160 centimeters and an area of 1500 square centimeters. What are the length and width of the rectangle?

13. The length of the diagonal between opposite corners of a rectangle is 20 inches and its length is 4 inches longer than its width. What is the perimeter of the rectangle?

14. The length of the diagonal between opposite corners of a rectangle is twice its width and its length is 27 feet. What is the area of the rectangle?

15. A fence costing $20 per foot is purchased to enclose a rectangular field whose length is 4 feet longer than its width and whose area is 437 square feet. What is the total cost of the fence?

16. A sail is in the form of a right triangle with the vertical leg 4 feet longer than the horizontal leg and with hypotenuse $4\sqrt{13}$ feet. What is the cost of the material used to make the sail if this material costs $10 per square foot?

Chapter 9. Systems of Linear Equations

9.1. Solve Systems of Linear Equations by Graphing

KYOTE Standards: *CR 21; CA 13*

In this section we discuss how to solve systems of two linear equations in two variables by graphing.

We saw in Section 7.2 that the graph of a linear equation in two variables is a line in the plane consisting of all points that satisfy the linear equation. A *solution* to a system of two linear equations in two variables is a point that satisfies *both* linear equations and is therefore a point on *both* lines. For example, consider the following system of equations.

$x+4y=5$ Equation of line 1
$2x-y=-8$ Equation of line 2

The point $(-3,2)$ satisfies the first equation since $-3+4(2)=5$, and so $(-3,2)$ is a point on line 1. The point $(-3,2)$ also satisfies the second equation since $2(-3)-2=-8$, and so $(-3,2)$ is a point on line 2. Therefore $(-3,2)$ is a *solution* to the system of equations because it satisfies *both* equations and is a *point* on both lines.

The graphical method of solving a system of two equations in two variables involves graphing the line corresponding to each equation and finding a point that is on both these lines. Example 1 illustrates this process.

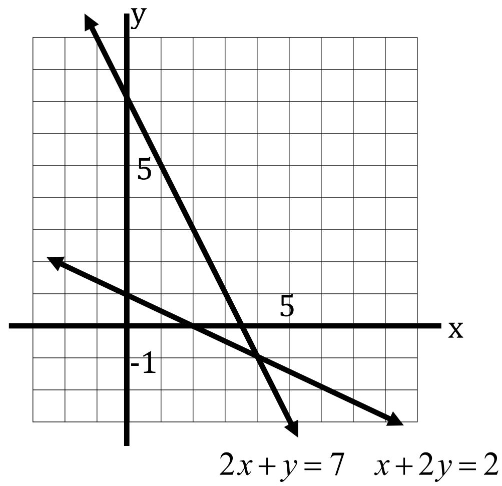
Example 1. Solve the system by graphing.

$2x+y=7$ First equation
$x+2y=2$ Second equation

Solution. We find the points corresponding to the x- and y-intercepts of both lines, together with an additional point on each line, and put them in the tables below.

$2x+y=7$		$x+2y=2$	
x	y	x	y
0	7	0	1
7/2	0	2	0
2	3	1	1/2

We graph the two lines on the same axes below.



We are looking for a pair of numbers (x, y) that satisfy *both* equations. Graphically, this means that we are looking for a point (x, y) that lies on *both* lines. There is clearly exactly one such point and it appears from the graph that this point has coordinates $x=4$ and $y=-1$. We substitute these values into both equations to verify that $x=4$ and $y=-1$ satisfy them both.

Therefore $x=4$ and $y=-1$ is the *unique* solution to the system.

Graphing is not an effective method for solving systems of linear equations. It is all but impossible, for example, to identify the exact solution of a system if the coordinates of that solution are not integers.

However, graphing is an excellent way to help us visualize a solution to a system of equations. For a system of two linear equations in two variables, it enables us to see immediately that there are only three possibilities:

1. The lines defined by the two equations in the system are not parallel and hence must intersect in exactly one point. In this case, the system of equations has a *unique (one and only one) solution*.
2. The lines defined by the two equations in the system are parallel and distinct and hence cannot intersect. In this case, the system of equations has *no solution*.
3. The lines defined by the two equations in the system are same line. In this case, the system of equations has *infinitely many solutions*. (Any pair of numbers that satisfy one of the equations also satisfies the other equation and is therefore a solution to the system of equations.)

Exercise Set 9.1

Solve the given system of equations by graphing. Sketch the graph of each equation in the system on the same set of axes. Determine whether the system has a unique solution, no solution or infinitely many solutions. If it has a unique solution, estimate that solution from the graph and determine whether your estimate is correct.

1. $\begin{aligned} x - y &= 2 \\ x + y &= 4 \end{aligned}$

2. $\begin{aligned} 2x + 3y &= 10 \\ x - y &= 0 \end{aligned}$

3. $\begin{aligned} 3x + 4y &= 12 \\ 3x + 4y &= 24 \end{aligned}$

4. $\begin{aligned} x + y &= 1 \\ -x + y &= 3 \end{aligned}$

5. $\begin{aligned} 2x + y &= 6 \\ x + 2y &= 6 \end{aligned}$

6. $\begin{aligned} x - 2y &= 4 \\ -2x + 4y &= -8 \end{aligned}$

7. $\begin{aligned} x - y &= 3 \\ x + 3y &= 7 \end{aligned}$

8. $\begin{aligned} y &= 2x + 1 \\ y &= 2x - 1 \end{aligned}$

9. $\begin{aligned} y &= x + 1 \\ y &= -2x + 4 \end{aligned}$

10. $\begin{aligned} 3x + 2y &= 6 \\ x - y &= 2 \end{aligned}$

11. $\begin{aligned} 6x + 4y &= 12 \\ 9x + 6y &= 18 \end{aligned}$

12. $\begin{aligned} x + y &= 3 \\ x - 2y &= -9 \end{aligned}$

9.2. Solve Systems of Linear Equations Analytically

KYOTE Standards: *CR 21; CA 13*

In this section we discuss how to solve systems of two linear equations in two variables analytically using two different methods: *substitution* and *elimination*.

Substitution Method

Use the following three steps as your guide when solving a system of two linear equations in two variables by substitution.

Step 1. Choose one equation and solve for one of its variables in terms of the other variable.

Step 2. Substitute the expression you found in Step 1 into the other equation to obtain an equation in one variable. Then solve for that variable.

Step 3. Substitute the value of the variable found in Step 2 back into the expression found in Step 2 to solve for the remaining variable.

Example 1. Solve the system by substitution.

$3x+4y=2$	First equation
$2x-y=5$	Second equation

Solution. The method of substitution requires that we solve for one of the variables in one of the equations. The best choice is to solve for y in the second equation because this approach is the only one that does not introduce any fractions. We obtain

$$y=2x-5 \qquad \text{Step 1}$$

We substitute $y=2x-5$ into the first equation and solve for x to obtain

$3x+4y=2$	First equation	
$3x+4(2x-5)=2$	Substitute $y=2x-5$	
$3x+8x-20=2$	Expand	
$11x=22$	Collect like terms	
$x=2$	Divide by 11	Step 2

We substitute $x=2$ into the equation $y=2x-5$ to obtain

$$y=2(2)-5=-1 \qquad \text{Step 3}$$

Thus $x=2$ and $y=-1$ is the unique solution to the system. It is a good idea to substitute these numbers in to the original equations to check your answer.

$$3(2)+4(-1)=2 \quad \text{First equation}$$
$$2(2)-(-1)=5 \quad \text{Second equation}$$

Geometrically, this means that the point $(2,-1)$ is on both the lines $3x+4y=2$ and $2x-y=5$, and it is the *only* point on both these lines.

Elimination Method

Use the following three steps as your guide when solving a system of two linear equations in two variables by elimination.

Step A. Multiply at least one of the equations by a number so that when the two equations are added, one of the variables is eliminated.

Step B. Add the two equations to eliminate one variable, then solve for the remaining variable.

Step C. Substitute the value of the variable found in Step B back into one of the original equations, and solve for the remaining variable.

Example 2. Solve the system in Example 1 by elimination.

$$3x+4y=2 \quad \text{First equation}$$
$$2x-y=5 \quad \text{Second equation}$$

Solution. The method of elimination requires that we eliminate one of the variables by obtaining an equation involving only the other variable. The easiest approach in this case would be to eliminate y by multiplying the second equation by 4 and adding it to the first. We obtain

$3x+4y=2$	First equation	
$8x-4y=20$	$4\times$Second equation	Step A

We add these equations to eliminate y and solve for x.

$11x=22$	First equation $+\ 4\times$Second equation	
$x=2$	Solve for x	Step B

We substitute $x=2$ into the first equation and solve for y. We obtain

$3(2)+4y=2$	Substitute $x=2$ into $3x+4y=2$	
$4y=-4$	Subtract $3(2)=6$	
$y=-1$	Divide by 4	Step C

Thus $x=2$ and $y=-1$ is the unique solution to the system, confirming the solution we obtained in Example 1.

Example 3. Solve the system.

$7x+2y=8$	First equation
$3x-5y=21$	Second equation

Solution. We can choose the method we wish to use in this case. We cannot solve for either variable in either equation without introducing fractions. So we rule out using substitution and choose elimination. If we use elimination and want to avoid fractions, then we need to change both equations by multiplying each one by an appropriate number. We multiply the first equation by 5 and the second equation by 2, and then add the two resulting equations to eliminate y.

$35x+10y=40$	$5\times$First equation	
$6x-10y=42$	$2\times$Second equation	Step A

We add these equations to eliminate y and solve for x.

$41x=82$	$5\times$First equation+ $2\times$Second equation	
$x=2$	Divide by 41	Step B

We substitute $x=2$ into the first equation and solve for y. We obtain

$7(2)+2y=8$	Substitute $x=2$ into $7x+2y=8$	
$2y=-6$	Subtract $7(2)=14$	
$y=-3$	Divide by 2	Step C

Thus $x=2$ and $y=-3$ is the unique solution to the system.

Example 4. Solve the system.

$2x-3y=19$	First equation
$y=2x-1$	Second equation

Solution. The substitution method is the logical choice in this case since the second equation $y=2x-1$ already has y written in terms of x and hence we can skip Step 1.

We substitute $y=2x-1$ into the first equation and solve for x to obtain

$2x-3y=19$	First equation	
$2x-3(2x-1)=19$	Substitute $y=2x-1$	
$2x-6x+3=19$	Expand	
$-4x=16$	Collect like terms	
$x=-4$	Divide by -4	Step 2

We substitute $x=-4$ into the equation $y=2x-1$ to obtain

$y=2(-4)-1=-9$ Step 3

Thus $x=-4$ and $y=-9$ is the unique solution to the system.

Example 5. Solve the given system by graphing. Confirm your answer by solving the system analytically using either substitution or elimination.

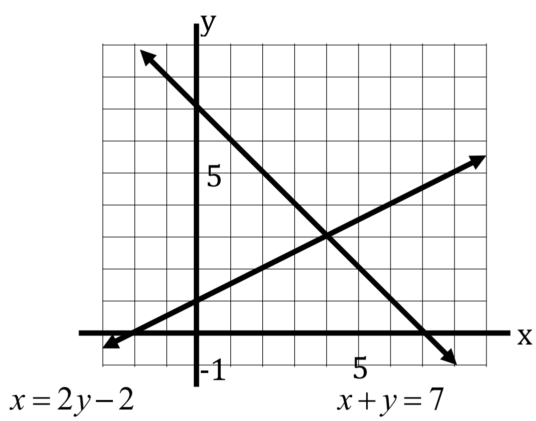
$x+y=7$ First equation

$x=2y-2$ Second equation

Solution. We find the points corresponding to the x- and y-intercepts of both lines, together with an additional point on each line, and put them in the tables below.

$x+y=7$		$x=2y-2$	
x	y	x	y
0	7	0	1
7	0	–2	0
2	5	2	2

We graph the two lines on the same axes below.



It appears from the graph that the two lines intersect at the point $(4,3)$ so that $x=4$ and $y=3$ is the unique solution to the system. We check this result by solving the system analytically. We choose the substitution method because the second equation $x=2y-2$ has x written in terms of y and so we can skip Step 1.

We substitute $x=2y-2$ into the first equation and solve for y to obtain

$x+y=7$ First equation

$2y-2+y=7$ Substitute $x=2y-2$

$3y=9$ Collect like terms

$y = 3$ Divide by 3 Step 2

We substitute $y = 3$ into the equation $x = 2y - 2$ to obtain

$x = 2(3) - 2 = 4$ Step 3

Thus $x = 4$ and $y = 3$ is the unique solution to the system and confirms what we obtained from the graph.

Example 6. Solve the given system.

(a) $x - 2y = -4$	**(b)** $x - 2y = -4$	First equation
$-3x + 6y = 17$	$-3x + 6y = 12$	Second equation

Solution. **(a)** We use the method of elimination to solve this system. We multiply the first equation by 3.

$3x - 6y = -12$ $3\times$First equation

$-3x + 6y = 17$ Second equation Step A

When we add these equations we discover that both x and y variables are eliminated and we obtain

$0 = 5$

What does this say about a solution to the original system? It says that *if* (x, y) is a solution to the system, *then* $0 = 5$, a statement that is false. We conclude that the system has *no solution*.

This conclusion is confirmed by noticing that the graphs of $x - 2y = -4$ and $-3x + 6y = 17$ are distinct parallel lines with slope $\frac{1}{2}$ and thus do not intersect.

(b) The system in part (b) is nearly the same as the system in part (a), but the solutions are much different. We again use the method of elimination and multiply the first equation by 3.

$3x - 6y = -12$ $3\times$First equation

$-3x + 6y = 12$ Second equation Step A

We add these equations and discover once again that both x and y variables are eliminated and we can write

$0 = 0$

This is a true statement but tells us nothing about the solution of the system. But if we examine the two equations $x-2y=-4$ and $-3x+6y=12$, we see that multiplying the first equation by -3 yields the second equation. In other words, these are two *different* equations for the *same* line.

Thus any pair of numbers (x,y) that satisfies one of these equations satisfies the other and is therefore a solution to the system. Consequently, this system has *infinitely many solutions*.

Exercise Set 9.2

Solve the given system by substitution.

1. $\begin{array}{l} x+2y=5 \\ y=2x+1 \end{array}$

2. $\begin{array}{l} 3x+2y=0 \\ x=y-5 \end{array}$

3. $\begin{array}{l} x=3y-7 \\ x=-y+1 \end{array}$

4. $\begin{array}{l} y=2x-3 \\ y=-3x+17 \end{array}$

5. $\begin{array}{l} x-y=3 \\ 3x+y=5 \end{array}$

6. $\begin{array}{l} x+y=8 \\ 3x-2y=-1 \end{array}$

7. $\begin{array}{l} 2x+y=5 \\ -x+3y=8 \end{array}$

8. $\begin{array}{l} 3x+4y=1 \\ 2x-y=19 \end{array}$

9. $\begin{array}{l} 2x+3y=10 \\ 2x+y=-6 \end{array}$

10. $\begin{array}{l} 3x+2y=5 \\ x+y=4 \end{array}$

11. $\begin{array}{l} 2x+4y=-9 \\ x-3y=8 \end{array}$

12. $\begin{array}{l} 3x-4y=3 \\ 6x-y=13 \end{array}$

Solve the given system by elimination.

13. $\begin{array}{l} x+2y=11 \\ 3x-2y=1 \end{array}$

14. $\begin{array}{l} 2x-y=4 \\ 2x+3y=-4 \end{array}$

15. $\begin{array}{l} x+y=7 \\ 2x-y=5 \end{array}$

16. $\begin{array}{l} 5x-3y=4 \\ -5x+7y=-16 \end{array}$

17. $\begin{aligned} 2x+3y&=-1 \\ 4x+2y&=2 \end{aligned}$

18. $\begin{aligned} 3x-4y&=6 \\ -9x+5y&=-18 \end{aligned}$

19. $\begin{aligned} 5x-3y&=0 \\ 3x+y&=14 \end{aligned}$

20. $\begin{aligned} x+5y&=1 \\ x-3y&=9 \end{aligned}$

21. $\begin{aligned} 3x-2y&=4 \\ 5x-3y&=7 \end{aligned}$

22. $\begin{aligned} 2x+3y&=12 \\ 7x-5y&=11 \end{aligned}$

23. $\begin{aligned} 4x-3y&=5 \\ 2x+6y&=5 \end{aligned}$

24. $\begin{aligned} 4x-10y&=-1 \\ 8x+5y&=8 \end{aligned}$

Solve the given system using either substitution or elimination.

25. $\begin{aligned} 4x-3y&=5 \\ x&=-1 \end{aligned}$

26. $\begin{aligned} 7x-2y&=15 \\ y&=-3 \end{aligned}$

27. $\begin{aligned} 6x-3y&=27 \\ x-8y&=12 \end{aligned}$

28. $\begin{aligned} -2x+5y-11&=0 \\ x-3y+6&=0 \end{aligned}$

29. $\begin{aligned} x+y-9&=0 \\ 4x-5y&=0 \end{aligned}$

30. $\begin{aligned} y&=-3x+12 \\ y&=2x+2 \end{aligned}$

31. $\begin{aligned} x&=7y+10 \\ x&=-2y+1 \end{aligned}$

32. $\begin{aligned} 2x-4y&=5 \\ 3x+5y&=2 \end{aligned}$

33. $\begin{aligned} 3x-8y+4&=0 \\ 6x+4y-7&=0 \end{aligned}$

34. $\begin{aligned} x&=2y-5 \\ 5x+6y+1&=0 \end{aligned}$

35. $\begin{aligned} y&=-5x+1 \\ 10x-3y&=7 \end{aligned}$

36. $\begin{aligned} x+y&=1 \\ 2x-y&=3 \end{aligned}$

37. $\begin{aligned} 2x-3y&=6 \\ -4x+6y&=9 \end{aligned}$

38. $\begin{aligned} 3x-5y&=0 \\ y&=\frac{3}{5}x+1 \end{aligned}$

39. $\begin{aligned} 3x-4y&=5 \\ -9x+12y&=-15 \end{aligned}$

40. $\begin{aligned} 3x+2y&=6 \\ y&=-\frac{3}{2}x+3 \end{aligned}$

Solve the given system by graphing. Confirm your answer by solving the system analytically using either substitution or elimination.

41. $\begin{aligned} x+y&=3 \\ 2x-y&=0 \end{aligned}$

42. $\begin{aligned} 2x-3y&=9 \\ x+2y&=1 \end{aligned}$

43. $\begin{aligned} y&=x+3 \\ y&=2x-1 \end{aligned}$

44. $\begin{aligned} 2x+y&=-1 \\ x-2y&=-8 \end{aligned}$

Chapter 10. Additional Topics

10.1. Functions.

KYOTE Standards: *CA 18*

The concept of a function is of central importance in mathematics. We have already done a great deal with functions in the text without using functional notation or defining the domain of a function. In this section we give a formal definition of a function, introduce functional notation, discuss the evaluation of a function at a number using functional notation, and discuss the domain of a function.

Definition 1. A *function* f is a rule that assigns to each element x in a set A exactly one element y in a set B. We denote that element y by $f(x)$. The set A is called the *domain* of f.

Note: The sets A and B in Definition 1 can be very general. In this section, however, we assume that A and B are sets of real numbers.

We can think of a function f as a machine that takes a number x in its domain as input and uses a specified rule to assign exactly one other number $f(x)$ as output.

Evaluation of a Function

We have evaluated a wide variety of algebraic expressions throughout the text, but we have not used functional notation to do so. For example, consider the quadratic function defined by

$$f(x) = x^2 - 3x$$

In this case, the rule f assigns to each real number x the real number $f(x)$ that is computed by squaring x and subtracting 3 times x. In particular, the function f assigns to the real number -2 the number $f(-2) = (-2)^2 - 3(-2) = 10$. Similarly, the function f assigns to the real number -1 the number $f(-1) = (-1)^2 - 3(-1) = 4$ and assigns to the real number $\frac{1}{2}$ the number $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) = -\frac{5}{4}$.

The next example provides additional explanation.

Example 1. Evaluate the function $f(x)=\dfrac{x^2}{x^3+1}$ at the values $x=-2,0,1,2,3$.

Solution. The function f takes any number it is given, squares that number, and then divides the result by the cube of the number plus 1. In other words, the function f takes any number it is given, substitutes that number for x in the rational expression $\dfrac{x^2}{x^3+1}$, and simplifies the result. We obtain the following:

Calculation	**Result**
$f(-2)=\dfrac{(-2)^2}{(-2)^3+1}=\dfrac{4}{-8+1}=-\dfrac{4}{7}$	$f(-2)=-\dfrac{4}{7}$
$f(0)=\dfrac{(0)^2}{(0)^3+1}=\dfrac{0}{1}=0$	$f(0)=0$
$f(1)=\dfrac{(1)^2}{(1)^3+1}=\dfrac{1}{2}$	$f(1)=\dfrac{1}{2}$
$f(2)=\dfrac{(2)^2}{(2)^3+1}=\dfrac{4}{8+1}=\dfrac{4}{9}$	$f(2)=\dfrac{4}{9}$
$f(3)=\dfrac{(3)^2}{(3)^3+1}=\dfrac{9}{27+1}=\dfrac{9}{28}$	$f(3)=\dfrac{9}{28}$

Domain of a Function

The domain of a function f, unless otherwise explicitly stated, is the set of all real numbers x for which $f(x)$ is also a real number. Thus the domain of any linear function, a line defined by $f(x)=mx+b$ with $m\neq 0$ and b real numbers, and any quadratic function, a parabola defined by $f(x)=ax^2+bx+c$ with $a\neq 0$, b and c real numbers, consists of all real numbers. More generally, the domain of any polynomial function consists of the set of all real numbers.

But you might have noticed that the rational function $f(x)=\dfrac{x^2}{x^3+1}$ in Example 1 cannot be evaluated at all values of x. The value $x=-1$ makes the denominator x^3+1 of this rational expression equal to 0. Thus $x=-1$ is *not* in the domain of this function and it is the only real number not in the domain.

The domain of f, denoted by $Domain(f)$, can be written in set notation:

$$Domain(f)=\{x\in R : x\neq -1\}$$

The set written as $\{x \in R : x \neq -1\}$ means *"the set of all x in R (the set of all real numbers) such that $x \neq -1$"*.

The domain of f can also be written in interval notation:

$$Domain(f) = (-\infty, -1) \cup (-1, \infty)$$

We discussed interval notation in Section 6.4. Recall that $(-\infty, -1)$ is the set of all real numbers less than -1 and $(-1, \infty)$ is the set of all real numbers greater -1. The union $(\cup)$ of these sets is all real numbers but -1.

We find the domains of three additional functions in the following examples.

Example 2. Find the domain of the function $f(x) = \dfrac{x}{x^2 - x - 6}$.

Solution. The domain of f consists of all real numbers x such that $f(x)$ is also a real number. We can evaluate f at any real number x unless the denominator of the rational expression $\dfrac{x}{x^2 - x - 6}$ is zero.

The denominator $x^2 - x - 6 = (x+2)(x-3)$ can be factored and so we see it is zero only when $x = -2$ and $x = 3$. Therefore every real number but -2 and 3 is in the domain of f. Thus the domain of f can be written in either set notation or in interval notation as follows:

$$Domain(f) = \{x \in R : x \neq -2, x \neq 3\}$$
$$Domain(f) = (-\infty, -2) \cup (-2, 3) \cup (3, \infty)$$

Set notation is preferred in this case because it is easier to read although interval notation is acceptable as well.

Example 3. Find the domain of the function $f(x) = \sqrt{2x - 3}$.

Solution. The domain of f consists of all real numbers x such that $f(x)$ is also a *real* number. The number 0 is *not* in the domain of f since $f(0) = \sqrt{-3}$ and the square root of a negative number *not* a real number. We see that x is in the domain of f if and only if the linear function $2x - 3$ under the square root sign is *not* a negative number. Therefore $2x - 3$ must be positive or zero and we can solve the inequality $2x - 3 \geq 0$ to find the domain of the function.

$2x - 3 \geq 0$	Given inequality
$2x \geq 3$	Add 3

$x \geq \frac{3}{2}$ Divide by 2

Thus the domain of f can be written in either set notation or in interval notation as follows:

$$Domain(f) = \left\{ x \in R : x \geq \frac{3}{2} \right\}$$
$$Domain(f) = \left[\frac{3}{2}, \infty \right)$$

Example 4. Find the domain of the function $f(x) = \frac{\sqrt{x+2}}{x-3}$.

Solution. In this example, we need to check that we are *not* taking the square root of a negative number *and* that we are *not* dividing by zero. Thus we are looking for real numbers x such that $x+2 \geq 0$ *and* $x-3 \neq 0$. We conclude that $x \geq -2$ *and* that $x \neq 3$ in order for x to be in the domain of f. Thus the domain of f can be written in either set notation or in interval notation as follows:

$$Domain(f) = \{ x \in R : x \geq -2 \text{ and } x \neq 3 \}$$
$$Domain(f) = [-2, 3) \cup (3, \infty)$$

Exercise Set 10.1

Evaluate the function at the indicated values.

1. $f(x) = x^2 - 2x + 5$; $x = -2, -1, 0, 1, 2$

2. $f(x) = x^3 + x$; $x = -2, -1, 0, 1, 2$

3. $f(x) = \sqrt[3]{x^2}$; $x = -2, -1, 0, 1, 2$

4. $f(x) = |1 - x|$; $x = -3, -1, 1, 3, 5$

5. $f(x) = \frac{1-2x}{1+2x}$; $x = -2, -1, 0, 1, 2$

6. $f(x) = \frac{|x+1|}{x+1}$; $x = -4, -3, -2, 0, 1, 2$

7. $f(x)=\sqrt{2x+1}$; $x=-\frac{1}{2},0,\frac{1}{2},\frac{3}{2},\frac{5}{2},4$

8. $f(x)=\frac{x^2-1}{x^2+1}$; $x=-2,-1,0,1,2$

9. $f(x)=\sqrt{x^2+x-2}$; $x=-5,-3,-2,1,2,5$

10. $f(x)=x+\frac{1}{x^2+1}$; $x=-2,-1,0,1,2$

11. $f(x)=\frac{1}{2x+1}+\frac{1}{2x-1}$; $x=-2,-1,0,1,2$

12. $f(x)=\sqrt{|x-1|}$; $x=-3,-2,-1,0,1,2,3$

Find the domain of the given function.

13. $f(x)=\frac{1}{x+4}$

14. $f(x)=\sqrt{x-2}$

15. $f(x)=x^2-4x+4$

16. $f(x)=\frac{x+2}{x^2-1}$

17. $f(x)=\frac{2x-4}{x^2+2x-15}$

18. $f(x)=\sqrt{5-x}$

19. $f(x)=|2x-3|$

20. $f(x)=\frac{x-1}{\sqrt{3x-1}}$

21. $f(x)=\frac{x^2+5x+6}{2x^2-x-1}$

22. $f(x)=\sqrt{|x-3|}$

23. $f(x)=\frac{1}{\sqrt{|x-3|}}$

24. $f(x)=\frac{x^2-1}{x^2+6x}$

25. $f(x)=\sqrt{3x-5}$

26. $f(x)=\frac{1}{\sqrt{3x-5}}$

27. $f(x)=\frac{1}{3x^2-x-14}$

28. $f(x)=\frac{\sqrt{x}+1}{\sqrt{x}-1}$

Find a function f with the given domain.

29. $Domain(f) = (-\infty, -2]$

30. $Domain(f) = [3, \infty)$

31. $Domain(f) = \{x \in R : x \neq 2\}$

32. $Domain(f) = \{x \in R : x \neq 2, x \neq -2\}$

33. $Domain(f) = \left[\frac{7}{3}, \infty\right)$

34. $Domain(f) = \{x \in R : x \neq 0, x \neq 5\}$

35. $Domain(f) = (2, \infty)$

36. $Domain(f) = [-2, 5) \cup (5, \infty)$

10.2. Solve Rational, Radical and Absolute Value Equations

KYOTE Standards: *CA 12*

An equation that involves rational expressions, square roots and absolute value expressions can often be reduced to a linear equation and solved. We consider three examples.

Example 1. Solve the equation $\frac{2}{x+1}+\frac{1}{x-1}=\frac{5}{x^2-1}$.

Solution. We clear fractions by multiplying both sides of the equation by the least common denominator of the fractions involved. (We can also use this technique to solve linear equations involving fractions.) To find the least common denominator, we first factor $x^2-1=(x+1)(x-1)$. We see that the least common denominator is $(x+1)(x-1)$. We then multiply both sides of the equation by the least common denominator and solve the resulting linear equation to obtain

$\frac{2}{x+1}+\frac{1}{x-1}=\frac{5}{x^2-1}$	Given equation
$(x+1)(x-1)\left(\frac{2}{x+1}+\frac{1}{x-1}\right)=(x+1)(x-1)\left(\frac{5}{x^2-1}\right)$	Multiply by $(x+1)(x-1)$
$2(x-1)+(x+1)=5$	Expand and simplify
$2x-2+x+1=5$	Expand
$3x=6$	Collect like terms
$x=2$	Divide by 3

Thus if this equation has a solution, then $x=2$ must be that solution. It is a good idea to check that $x=2$ is indeed a solution by substituting it into the original equation.

Example 2. Solve the equation $2\sqrt{2x-7}+5=11$.

Solution. We isolate the square root expression on one side of the equation, square both sides, and solve the resulting linear equation to obtain

$2\sqrt{2x-7}+5=11$	Given equation
$2\sqrt{2x-7}=6$	Subtract 5
$\sqrt{2x-7}=3$	Divide by 2
$2x-7=9$	Square both sides
$2x=16$	Add 7
$x=8$	Divide by 2

Thus if this equation has a solution, then $x=8$ must be that solution. It is a good idea to check that $x=8$ is indeed a solution by substituting it into the original equation.

Example 3. Solve the equation $|3x+5|=8$.

Solution. If the absolute value of a number is 8, then that number must be either 8 or -8. In this case, the number $3x+5$ must be either 8 or -8. Thus we must solve two linear equations: $3x+5=8$ and $3x+5=-8$. We obtain

$3x+5=8$	Given equation
$3x=3$	Subtract 5
$x=1$	Divide by 3

$3x+5=-8$	Given equation
$3x=-13$	Subtract 5
$x=-\frac{13}{3}$	Divide by 3

Thus $x=1$ and $x=-\frac{13}{3}$ are the two solutions. It is easy to check that these two numbers both satisfy the original equation $|3x+5|=8$.

Exercise Set 10.2

Solve the given equation.

1. $\frac{1}{x}=\frac{3}{x+4}$

2. $\sqrt{x-1}=2$

3. $\frac{1}{x}+\frac{1}{5x}=6$

4. $|3x|=12$

5. $\frac{\sqrt{2x-1}}{3}=1$

6. $\frac{1}{x+1}-\frac{1}{3}=\frac{1}{3x+3}$

7. $|2x-5|=7$

8. $3\sqrt{3x+1}-5=7$

9. $x-\frac{x}{3}=\frac{x}{2}+1$

10. $|x-4|=0.01$

11. $\sqrt{\frac{5-2x}{3}}=2$

12. $\frac{1}{x}-\frac{1}{2}=\frac{3}{4}$

13. $\frac{1}{2x-1}+\frac{5}{2}=\frac{3}{4x-2}$

14. $\left|\frac{x-3}{5}\right|=2$

15. $4\sqrt{7-3x}=12$

16. $\dfrac{2}{x-1}+\dfrac{1}{x^2+x-2}=\dfrac{1}{x+2}$

17. $|9-2x|=7$

18. $\sqrt{|2x+6|}=4$

19. $\dfrac{3}{x}-\dfrac{2}{x+1}=\dfrac{1}{x^2+x}$

20. $\dfrac{1}{2}x-\dfrac{x+4}{3}=1$

21. $\sqrt{|3x-6|}=3$

22. $\dfrac{3}{2x+1}=\dfrac{4}{5x}$

Appendix 1

KYOTE College Readiness Placement Exam Standards

1. Evaluate numerical expressions involving signed numbers, positive integer exponents, order of operations and parentheses.

2. Evaluate algebraic expressions at specified values of their variables using signed numbers, positive integer exponents, order of operations and parentheses.

3. Perform arithmetic calculations involving fractions, decimals and percents.

4. Order fractions and decimals on a number line.

5. Solve applied arithmetic problems using appropriate units, including problems involving percentage increase and decrease, rates and proportions.

6. Solve simple geometry problems involving properties of rectangles and triangles.

7. Solve simple coordinate geometry problems.

8. Add and subtract polynomials.

9. Multiply polynomials.

10. Simplify algebraic expressions involving positive and negative integer exponents, and square roots.

11. Factor a polynomial in one or more variables by factoring out its greatest common factor; factor a quadratic polynomial.

12. Add, subtract and multiply simple rational expressions.

13. Simplify a rational expression.

14. Solve a linear equation.

15. Solve a multivariable equation for one of its variables.

16. Use a linear equation to solve a simple word problem.

17. Solve a linear inequality.

18. Find the slope of a line given two points on the line or its equation; find the equation of a line given two points on the line, or a point on the line and the slope of the line.

19. Graph a line given its equation; find the equation of a line given its graph.

20. Solve a quadratic equation by factoring or by using the quadratic formula.

21. Solve a system of two linear equations in two variables.

KYOTE College Algebra Placement Exam Standards

1. Evaluate algebraic expressions at specified values of their variables using signed numbers, rational exponents, order of operations and parentheses.

2. Add, subtract and multiply polynomials.

3. Simplify algebraic expressions involving integer exponents.

4. Simplify algebraic expressions involving square roots and cube roots.

5. Factor a polynomial in one or more variables by factoring out its greatest common factor. Factor a trinomial. Factor the difference of squares.

6. Add, subtract, multiply and divide rational expressions.

7. Simplify a rational expression.

8. Solve a linear equation.

9. Solve a multivariable equation for one of its variables.

10. Solve a linear inequality in one variable.

11. Solve a quadratic equation.

12. Solve an equation involving a radical, a rational or an absolute value expression.

13. Solve a system of two linear equations in two variables.

14. Solve problems that can be modeled using a linear or quadratic equation or expression.

15. Solve geometry problems using the Pythagorean theorem and the properties of similar triangles.

16. Understand and apply the relationship between the properties of a graph of a line and its equation.

17. Find the intercepts and the graph of a parabola given its equation. Find an equation of a parabola given its graph.

18. Evaluate a function at a number in its domain. Find the domain of a rational function or the square root of a linear function.

Appendix 2

Kentucky College Readiness Indicators and Learning Outcomes

College Readiness Indicators [1,2]

Beginning fall 2012, all public postsecondary institutions in Kentucky will use the following benchmarks as college readiness indicators. Upon admission to a public postsecondary institution, students scoring at or above the scores indicated will not be required to complete developmental, supplemental, or transitional coursework and will be allowed entry into college credit-bearing coursework that counts toward degree credit requirements.

Readiness Score Area	ACT Score	SAT Score	COMPASS	KYOTE
English (Writing)	English 18 or higher	Writing 430 or higher	Writing 74 or higher [3,4]	6 or higher[5]
Reading	Reading 20 or higher	Critical Reading 470 or higher	Reading 85 or higher[6]	20 or higher
Mathematics (General Education, Liberal Arts Courses)	Mathematics 19 or higher	Mathematics 460 or higher	Algebra Domain 36 or higher[7]	College Readiness Mathematics 22 or higher
Mathematics (College Algebra)	Mathematics 22 or higher	Mathematics 510 or higher	Algebra Domain 50 or higher[8]	College Algebra 14 or higher[9]
Mathematics (Calculus)	Mathematics 27 or higher	Mathematics 610 or higher	NA[10]	Calculus TBA

1. Institutional admission policies are comprised of many factors including, but not limited to high school completion or a general education equivalency diploma (GED), high school coursework, ACT or SAT scores, high school GPA, class rank, an admission essay or interview, submission of an academic and/or civic activity portfolio, etc. Placement exam results are used for course placement after a student is admitted to a postsecondary institution.
2. A COMPASS or KYOTE placement test score will be guaranteed as an indicator of college readiness for 12 months from the date the placement exam is administered.
3. An Asset writing score of 43 or higher indicates readiness. Asset is the paper-pencil version of COMPASS.
4. COMPASS E-Write scores of 9 on a 12 point scale or 6 on an 8 point scale indicate readiness.

5. A common rubric will be used to score the KYOTE Writing Essay. The rubric has an eight point scale. A score of 6 is needed to demonstrate readiness.
6. An Asset reading score of 44 or higher indicates readiness. Asset is the paper-pencil version of COMPASS.
7. An Asset Elementary Algebra Score of 41 or an Intermediate Algebra score of 39 indicates readiness for a general education course, typically in the social sciences.
8. An Asset elementary algebra score of 46 or an intermediate algebra score of 43 indicates readiness for college algebra.
9. For the 2011-12 school year a KYOTE College Readiness Mathematics Placement score of 27 or higher will be used to indicate readiness for College Algebra. For the 2012-13 and beyond, only the KYOTE College Algebra placement test score of 14 or higher will be used to indicate readiness for College Algebra.
10. There is not a COMPASS or Asset indicator for Calculus readiness.

Learning Outcomes for Transitional Mathematics Courses

Required for Placement into a College Level Mathematics Course

MATHEMATICS FOR THE LIBERAL ARTS

Transitional, developmental, and supplemental education mathematics courses objectives for a liberal arts mathematics course:

1. Perform exact arithmetic calculations involving fractions, decimals and percents.
2. Simplify and evaluate algebraic expressions using the order of operations.
3. Use the properties of integer exponents and rational exponents of the form 1/n.
4. Calculate and solve applied problems of the perimeter, circumference, area, volume, and surface area.
5. Solve proportions.
6. Determine the slope of a line given two points, its graph, or its equation; determine an equation of a line given two points or a point and slope.
7. Solve and graph linear equations and inequalities in one and two variables.
8. Simplify square roots of algebraic and numerical expressions.
9. Solve systems of two linear equations in two variables.
10. Graph parabolas on the rectangular coordinate system.
11. Solve quadratic equations.
12. Factor the greatest common factor from a quadratic; factor simple trinomial of the form $ax^2 + bx + c$.
13. Add, subtract, and multiply polynomials with one or more variables.
14. Solve applied problems using the above competencies.
15. Recommendation for inclusion: Apply the concepts in the course to model and solve applications based on linear and quadratic functions.

Students successfully completing the liberal arts mathematics course may need to complete an additional transitional course to prepare for college algebra.

Courses from public postsecondary institutions that meet the mathematics readiness learning outcomes for a liberal arts mathematics course:

KCTCS—MAT 120
Eastern Kentucky University—MAT 095
Kentucky State University—MAT 096
Morehead State University—MATH 091
Murray State University—MAT 100
Northern Kentucky University—MAHD 095
Western Kentucky University—DMA 096
University of Kentucky
University of Louisville

Learning Outcomes for Transitional Mathematics Courses

Required for Placement into College Algebra

COLLEGE ALGEBRA

Transitional, developmental, and supplemental education mathematics courses objectives for college algebra:

1. Add, subtract, multiply, and divide polynomials.
2. Factor polynomials including finding the greatest common factor, using grouping, recognizing special products, and factoring general trinomials.
3. Use the properties of rational exponents.
4. Add, subtract, multiply, and divide rational expressions.
5. Solve quadratic equations using factoring, completing the square, and the quadratic formula.
6. Solve polynomial and rational equations.
7. Solve systems of linear equations in two unknowns.
8. Solve absolute value equations and solve and graph absolute value inequalities.
9. Solve and graph linear equations and inequalities in one or two variables.
10. Solve equations with radicals.
11. Introduce complex numbers.
12. Evaluate real numbers raised to rational exponents and simplify expressions containing rational exponents.
13. Convert expressions with rational exponents to radical form and vice versa.
14. Understand the concept of slope, how it relates to graphs, and its relation to parallel and perpendicular lines.
15. Determine an equation of a line given two points, a point, and slope, a point and a parallel or perpendicular line.
16. Determine whether a given correspondence or graph represents a function.
17. Evaluate functions and find the domains of polynomial, rational, and square root functions.
18. Graph parabolas by finding the vertex and axis of symmetry and plotting points.
19. Apply the concepts in the course to model and solve applications based on linear, quadratic, and exponential functions.

Courses from public postsecondary institutions that meet the mathematics readiness learning outcomes for college algebra:

KCTCS—MAT 120
Eastern Kentucky University—MAT 097 or MAT 098
Kentucky State University—MAT 097
Morehead State University—MATH 093
Murray State University—MAT 105
Northern Kentucky University—MAHD 099
Western Kentucky University—DMA 096
University of Kentucky
University of Louisville